

# CONVEX COCOMPACTNESS IN PSEUDO-RIEMANNIAN HYPERBOLIC SPACES

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*To Bill Goldman on his 60th birthday*

**ABSTRACT.** Anosov representations of word hyperbolic groups into higher-rank semisimple Lie groups are representations with finite kernel and discrete image that have strong analogies with convex cocompact representations into rank-one Lie groups. However, the most naive analogy fails: generically, Anosov representations do not act properly and cocompactly on a convex set in the associated Riemannian symmetric space. We study representations into projective indefinite orthogonal groups  $\mathrm{PO}(p, q)$  by considering their action on the associated pseudo-Riemannian hyperbolic space  $\mathbb{H}^{p, q-1}$  in place of the Riemannian symmetric space. Following work of Barbot and Mériçot in anti-de Sitter geometry, we find an intimate connection between Anosov representations and the natural notion of convex cocompactness in this setting.

## 1. INTRODUCTION

Convex cocompact subgroups of rank-one semisimple Lie groups are an important class of discrete groups whose actions on the associated Riemannian symmetric space (and its visual boundary at infinity) exhibit many desirable geometric and dynamical properties. Their study has been particularly important in the setting of Kleinian groups and hyperbolic geometry. This paper studies a generalized notion of convex cocompactness in the higher-rank setting of projective indefinite orthogonal groups  $\mathrm{PO}(p, q)$ , described in terms of the action on the projective space  $\mathbb{P}(\mathbb{R}^{p, q})$  and on the associated pseudo-Riemannian hyperbolic space  $\mathbb{H}^{p, q-1}$ . Our forthcoming paper [DGK2] will extend many of these ideas to the setting of discrete subgroups of the projective general linear group  $\mathrm{PGL}(\mathbb{R}^n)$  which do not necessarily preserve any quadratic form.

**1.1. Convex cocompactness in projective orthogonal groups.** In the whole paper, we fix integers  $p, q \in \mathbb{N}^*$  with  $p + q \geq 3$  and let  $G = \mathrm{PO}(p, q)$  be the orthogonal group, modulo its center  $\{\pm I\}$ , of a nondegenerate symmetric

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bilinear form  $\langle \cdot, \cdot \rangle_{p,q}$  of signature  $(p, q)$  on  $\mathbb{R}^{p+q}$ . We denote by  $\mathbb{R}^{p,q}$  the space  $\mathbb{R}^{p+q}$  endowed with the symmetric bilinear form  $\langle \cdot, \cdot \rangle_{p,q}$ . For any linear subspace  $W$  of  $\mathbb{R}^{p,q}$ , we denote by  $W^\perp$  the orthogonal of  $W$  for  $\langle \cdot, \cdot \rangle_{p,q}$ . We use similar notation in  $\mathbb{P}(\mathbb{R}^{p,q})$ : in particular, for  $z \in \mathbb{P}(\mathbb{R}^{p,q})$  the set  $z^\perp$  is a projective hyperplane of  $\mathbb{P}(\mathbb{R}^{p,q})$ , which contains  $z$  if and only if  $\langle z, z \rangle_{p,q} = 0$ .

When  $q = 1$ , the group  $G$  is the group of isometries of the real hyperbolic space

$$\mathbb{H}^p = \{[x] \in \mathbb{P}(\mathbb{R}^{p,1}) \mid \langle x, x \rangle_{p,1} < 0\},$$

which is also the Riemannian symmetric space  $G/K$  associated with  $G$ . Recall that a discrete subgroup  $\Gamma$  of  $G = \mathrm{PO}(p, 1)$  is said to be *convex cocompact* if it acts cocompactly on some nonempty closed convex subset  $\mathcal{C}$  of  $\mathbb{H}^p$ . Note that since  $\Gamma$  is discrete and  $\mathbb{H}^p$  is Riemannian, the action is automatically properly discontinuous, and so the quotient  $\Gamma \backslash \mathcal{C}$  is a hyperbolic orbifold, or a manifold if the action is free, with convex boundary. Basic examples of convex cocompact subgroups include uniform lattices of  $G$  and Schottky subgroups of  $G$  playing ping pong on  $\partial_\infty \mathbb{H}^p$ . In the case  $p = 3$ , for which the accidental isomorphism  $\mathrm{PO}(3, 1)_0 \simeq \mathrm{PSL}_2(\mathbb{C})$  makes  $G$  a complex group, the realm of Kleinian groups gives an abundance of interesting examples coming both from complex analysis à la Ahlfors and Bers and from three-manifold topology and Thurston's geometrization program. Notable are the quasi-Fuchsian groups (isomorphic to closed surface groups) which are deformations of Fuchsian surface subgroups of  $\mathrm{PO}(2, 1) \subset \mathrm{PO}(3, 1)$ .

Assume  $\mathrm{rank}_{\mathbb{R}} := \min(p, q) \geq 2$  and let  $K = \mathrm{P}(\mathrm{O}(p) \times \mathrm{O}(q))$  be a maximal compact subgroup of  $G$ . The group  $G$  is the isometry group of the Riemannian symmetric space  $G/K$ , and it is natural to study the discrete subgroups  $\Gamma$  of  $G$  that act cocompactly on some convex subset of  $G/K$ . This naive generalization of convex cocompactness turns out to be quite restrictive due to the following general result proved independently by Kleiner–Leeb [KL] and Quint [Q].

**Fact 1.1** ([KL, Q]). *Let  $G$  be a real semisimple Lie group of real rank  $\geq 2$  and  $K$  a maximal compact subgroup of  $G$ . Any Zariski-dense discrete subgroup of  $G$  acting properly discontinuously and cocompactly on some nonempty closed convex subset  $\mathcal{C}$  of the Riemannian symmetric space  $G/K$  is a uniform lattice in  $G$ .*

In this paper, we propose instead a notion of convex cocompactness in  $G = \mathrm{PO}(p, q)$  in terms of the action on the real projective space  $\mathbb{P}(\mathbb{R}^{p,q})$ , and in particular on the invariant open domain

$$\mathbb{H}^{p,q-1} = \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x, x \rangle_{p,q} < 0\} \simeq G/\mathrm{O}(p, q-1)$$

which is the projective model for a pseudo-Riemannian symmetric space associated to  $G$ . Indeed,  $G$  is the isometry group for the pseudo-Riemannian structure on  $\mathbb{H}^{p,q-1}$  of signature  $(p, q-1)$  induced by the symmetric bilinear form  $\langle \cdot, \cdot \rangle_{p,q}$  (see Section 2.1). Geodesics of  $\mathbb{H}^{p,q-1}$  are intersections of  $\mathbb{H}^{p,q-1}$  with straight lines of  $\mathbb{P}(\mathbb{R}^{p,q})$ . For  $q = 1$ , the space  $\mathbb{H}^{p,q-1}$  is the real

hyperbolic space  $\mathbb{H}^p$  in its projective model. In general,  $\mathbb{H}^{p,q-1}$  is a pseudo-Riemannian analogue of  $\mathbb{H}^p$ , with constant negative sectional curvature.

Recall that a subset of projective space is said to be *convex* (resp. *properly convex*) if it is convex (resp. convex and bounded) in some affine chart. Unlike real hyperbolic space, for  $q > 1$  the space  $\mathbb{H}^{p,q-1}$  is *not* a convex subset of the projective space  $\mathbb{P}(\mathbb{R}^{p,q})$ , and the basic operation of taking convex hulls is not well defined. Nonetheless, a subset of  $\mathbb{H}^{p,q-1}$  may be called *convex* if it is convex as a subset of projective space or, from an intrinsic point of view, if any two points of the subset are connected by a unique geodesic segment inside the subset. It may be called *properly convex* if it is properly convex as a subset of projective space or, from an intrinsic point of view, if its closure in  $\mathbb{H}^{p,q-1}$  is convex.

For  $q = 2$ , the Lorentzian space  $\mathbb{H}^{p,q-1}$  is the  $(p+1)$ -dimensional *anti-de Sitter space*  $\text{AdS}^{p+1}$ , where a notion of *AdS quasi-Fuchsian group* has been studied by Mess [Me] and Barbot–Mérigot [BM, Ba]. Inspired by this notion, we make the following definition.

**Definition 1.2.** An irreducible discrete subgroup  $\Gamma$  of  $G = \text{PO}(p, q)$  is  $\mathbb{H}^{p,q-1}$ -*convex cocompact* if it acts properly discontinuously and cocompactly on some nonempty, closed properly convex subset  $\mathcal{C}$  of  $\mathbb{H}^{p,q-1}$ .

Here we say that  $\Gamma$  is *irreducible* if it does not preserve any projective subspace of  $\mathbb{P}(\mathbb{R}^{p,q})$  of positive codimension. We note that an  $\mathbb{H}^{p,q-1}$ -convex cocompact group is always finitely generated.

Note that a discrete subgroup  $\Gamma$  of  $\text{PO}(p, q)$  need not act properly discontinuously on  $\mathbb{H}^{p,q-1}$ . When  $\Gamma$  preserves a properly convex subset  $\mathcal{C} \subset \mathbb{H}^{p,q-1}$ , the action on the interior of  $\mathcal{C}$  is properly discontinuous (see Section 2.3), but the action on  $\mathcal{C}$  need not be. The requirement of proper discontinuity in Definition 1.2 guarantees that the accumulation points of any orbit are contained in the *ideal boundary* of  $\mathcal{C}$ , defined to be  $\partial_i \mathcal{C} := \overline{\mathcal{C}} \setminus \mathcal{C}$ , where  $\overline{\mathcal{C}}$  is the closure of  $\mathcal{C}$  in  $\mathbb{P}(\mathbb{R}^{p,q})$ . Since  $\mathcal{C}$  is assumed to be closed in  $\mathbb{H}^{p,q-1}$ , the ideal boundary  $\partial_i \mathcal{C}$  is contained in the *boundary* of  $\mathbb{H}^{p,q-1}$ , namely

$$\partial_{\mathbb{P}} \mathbb{H}^{p,q-1} := \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x, x \rangle_{p,q} = 0\}.$$

Recall that any irreducible discrete subgroup  $\Gamma$  of  $\text{PGL}(\mathbb{R}^n)$  preserving a properly convex subset of  $\mathbb{P}(\mathbb{R}^n)$  contains a *proximal* element, i.e. an element with a unique attracting fixed point in  $\mathbb{P}(\mathbb{R}^n)$  (see [B2, Prop. 3.1]). The *limit set*  $\Lambda_{\Gamma} \subset \mathbb{P}(\mathbb{R}^n)$  (Definition 2.3) is the closure of the attracting fixed points of proximal elements of  $\Gamma$ . If  $\Gamma$  is contained in  $G = \text{PO}(p, q)$ , then  $\Lambda_{\Gamma}$  is contained in  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  (see Remark 2.4). For  $\Gamma$  and  $\mathcal{C}$  as in Definition 1.2, the set  $\mathcal{C}$  must have nonempty interior and so  $\Lambda_{\Gamma} \subset \partial_i \mathcal{C}$ .

**Remark 1.3.** The boundary  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  divides  $\mathbb{P}(\mathbb{R}^{p,q})$  into two connected components. One component is  $\mathbb{H}^{p,q-1}$  and the other is

$$\mathbb{S}^{p-1,q} = \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x, x \rangle_{p,q} > 0\},$$

which inherits from  $\langle \cdot, \cdot \rangle_{p,q}$  a pseudo-Riemannian metric of positive curvature. However, multiplication by  $-1$  transforms  $\langle \cdot, \cdot \rangle_{p,q}$  into a form of signature  $(q, p)$ , and  $\mathbb{S}^{p-1,q}$  into the copy of  $\mathbb{H}^{q,p-1}$  defined by  $-\langle \cdot, \cdot \rangle_{p,q}$ . Note that  $\mathrm{PO}(\langle \cdot, \cdot \rangle_{p,q}) = \mathrm{PO}(-\langle \cdot, \cdot \rangle_{p,q}) \simeq \mathrm{PO}(q, p)$ . Hence, rather than study two very similar notions of convex cocompactness in pseudo-Riemannian hyperbolic spaces  $\mathbb{H}^{p,q-1}$  and pseudo-Riemannian “spheres”  $\mathbb{S}^{p-1,q}$ , we will use the above isomorphism to exchange  $\mathbb{S}^{p-1,q}$  with  $\mathbb{H}^{q,p-1}$  when convenient.

**1.2. Goals of the paper.** There are three main goals. First, we show that the notion of convex cocompact subgroup in our setting is closely related to the notion of Anosov representation — a notion which has become fundamental in the study of higher Teichmüller theory since it was introduced by Labourie [L]. Second, we show that in the setting of discrete irreducible subgroups of  $\mathrm{PO}(p, q)$ , our notion of convex cocompact subgroup is equivalent to a notion of strong projective convex cocompactness introduced by Crampon–Marquis [CM]. Third, we show that a natural construction of Coxeter groups in projective orthogonal groups going back to Tits gives rise to many examples of convex cocompact subgroups in  $\mathbb{H}^{p,q-1}$  and therefore to many new examples of Anosov representations and strongly projectively convex cocompact subgroups of  $\mathrm{PO}(p, q) \subset \mathrm{PGL}(\mathbb{R}^{p+q})$ .

**1.3. Link with Anosov representations.** The main result of this paper is to give a close connection between convex cocompact actions and Anosov representations. These representations were introduced by Labourie [L] for fundamental groups of compact negatively-curved manifolds, and generalized for arbitrary word hyperbolic groups by Guichard–Wienhard [GW]. They have been extensively studied recently by many authors (see e.g. [BCLS, KLPb, GGKW, BPS]) and now play a crucial role in higher Teichmüller theory (see e.g. [BIW2]).

Let  $P_1^{p,q}$  be the stabilizer in  $G = \mathrm{PO}(p, q)$  of an isotropic line of  $\mathbb{R}^{p,q}$ ; it is a parabolic subgroup of  $G$ , and  $G/P_1^{p,q}$  identifies with the boundary  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  of  $\mathbb{H}^{p,q-1}$ . By definition, a  $P_1^{p,q}$ -Anosov representation of a word hyperbolic group  $\Gamma$  into  $G$  is a representation  $\rho : \Gamma \rightarrow G$  for which there exists a continuous,  $\rho$ -equivariant boundary map  $\xi : \partial_{\infty}\Gamma \rightarrow \partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  which

- (i) is transverse (a strengthening of injectivity), meaning that  $\xi(\eta) \notin \xi(\eta')^{\perp}$  for any  $\eta \neq \eta'$  in  $\partial_{\infty}\Gamma$ ,
- (ii) has an associated flow with some uniform contraction/expansion properties described in [L, GW].

As a consequence of (ii), every infinite-order element of  $\Gamma$  is proximal in  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  and the limit set  $\Lambda_{\Gamma}$  (Definition 2.3 and Remark 2.4) coincides with the image  $\xi(\partial_{\infty}\Gamma)$  of the boundary map. By [GW, Prop. 4.10], if  $\rho(\Gamma)$  is irreducible then condition (ii) is automatically satisfied as soon as (i) is.

In real rank 1, it is easy to see [GW, Th. 5.15] that a discrete subgroup  $\Gamma$  of  $G = \mathrm{PO}(p, 1)$  is convex cocompact if and only if  $\Gamma$  is word hyperbolic

and the natural inclusion  $\Gamma \hookrightarrow G$  is  $P_1^{p,1}$ -Anosov. In this paper, we prove the following generalization to higher real rank.

**Theorem 1.4.** *For  $p, q \in \mathbb{N}^*$  with  $p + q \geq 3$ , let  $\Gamma$  be an irreducible discrete subgroup of  $G = \mathrm{PO}(p, q)$ .*

- (1) *If  $\Gamma$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact, then it is word hyperbolic and the natural inclusion  $\Gamma \hookrightarrow G$  is  $P_1^{p,q}$ -Anosov.*
- (2) *Conversely, if  $\Gamma$  is word hyperbolic with connected boundary  $\partial_\infty \Gamma$  and if the natural inclusion  $\Gamma \hookrightarrow G$  is  $P_1^{p,q}$ -Anosov, then  $\Gamma$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact or  $\mathbb{H}^{q,p-1}$ -convex cocompact (after identifying  $\mathrm{PO}(p, q)$  with  $\mathrm{PO}(q, p)$ , see Remark 1.3).*

*If these conditions are satisfied, then the ideal boundary  $\partial_i \mathcal{C}$  of any closed, properly convex subset  $\mathcal{C} \subset \mathbb{H}^{p,q-1}$  with a proper and cocompact action of  $\Gamma$  as in Definition 1.2 is the limit set  $\Lambda_\Gamma \subset \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ .*

**Remark 1.5.** The special case when  $q = 2$  and  $\Gamma$  is the fundamental group of a closed hyperbolic  $p$ -manifold follows from work of Mess [Me] for  $p = 2$  and work of Barbot–Mérigot [BM] for  $p \geq 3$ .

**1.4. Anosov representations with negative or positive limit set.** We may replace the connectedness assumption of Theorem 1.4.(2) with the following simple consistency condition on the image of the boundary map.

**Definition 1.6.** A subset  $\Lambda$  of  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  is *negative* (resp. *positive*) if it lifts to a cone of  $\mathbb{R}^{p,q} \setminus \{0\}$  on which all inner products  $\langle \cdot, \cdot \rangle_{p,q}$  of noncolinear points are negative (resp. positive). Equivalently (Lemma 3.2 and Remark 3.4), every triple of distinct points of  $\Lambda$  spans a triangle fully contained in  $\mathbb{H}^{p,q-1}$  (resp.  $\mathbb{S}^{p-1,q}$ ) outside of the vertices.

By a *cone* we mean a subset of  $\mathbb{R}^{p,q} \setminus \{0\}$  which is invariant under multiplication by positive scalars. Recall from Remark 1.3 that  $\mathbb{H}^{p,q-1}$  and  $\mathbb{S}^{p-1,q}$  are the two connected components of  $\mathbb{P}(\mathbb{R}^{p,q}) \setminus \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ . In the Lorentzian setting (i.e.  $q = 2$ ), a negative subset is also called an *acausal* subset.

Since connectedness of the set of unordered distinct triples of  $\Lambda$  follows from connectedness of  $\Lambda$  (Fact A.1), the following holds (see Section 3.2).

**Proposition 1.7.** *If a closed subset  $\Lambda$  of  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  is connected and transverse, then it is negative or positive.*

Here we say that  $\Lambda$  is *transverse* if for any  $y \neq z$  in  $\Lambda$  we have  $y \notin z^\perp$ .

Theorem 1.4 is an immediate consequence of Proposition 1.7 and of the following, which is the main result of the paper.

**Theorem 1.8.** *Let  $p, q \in \mathbb{N}^*$  with  $p + q \geq 3$ . For an irreducible discrete subgroup  $\Gamma$  of  $G = \mathrm{PO}(p, q)$ , the following two conditions are equivalent:*

- (i)  *$\Gamma$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact,*
- (ii)  *$\Gamma$  is word hyperbolic, the natural inclusion  $\Gamma \hookrightarrow G$  is  $P_1^{p,q}$ -Anosov, and the limit set  $\Lambda_\Gamma \subset \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  is negative.*

Similarly, the following two conditions are equivalent:

- (iii)  $\Gamma$  is  $\mathbb{H}^{q,p-1}$ -convex cocompact (after identifying  $\mathrm{PO}(p, q)$  with  $\mathrm{PO}(q, p)$ ),
- (iv)  $\Gamma$  is word hyperbolic, the natural inclusion  $\Gamma \hookrightarrow G$  is  $P_1^{p,q}$ -Anosov, and the limit set  $\Lambda_\Gamma \subset \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  is positive.

By [L, GW], the space of  $P_1^{p,q}$ -Anosov representations is open in  $\mathrm{Hom}(\Gamma, G)$ . In fact, the space of  $P_1^{p,q}$ -Anosov representations with negative limit set is also open (Proposition 3.6), and so Theorem 1.8 has the following consequence.

**Corollary 1.9.** *Let  $p, q \in \mathbb{N}^*$  with  $p + q \geq 3$ . For a finitely generated group  $\Gamma$ , the set of irreducible injective representations  $\Gamma \rightarrow G = \mathrm{PO}(p, q)$  whose image is  $\mathbb{H}^{p,q-1}$ -convex cocompact is open in  $\mathrm{Hom}(\Gamma, G)$ .*

**Remark 1.10.** In the special case when  $p \geq q = 2$  (i.e.  $\mathbb{H}^{p,q-1}$  is the Lorentzian *anti-de Sitter space*  $\mathbb{H}^{p,1} = \mathrm{AdS}^{p+1}$ ) and  $\Gamma$  is isomorphic to the fundamental group of a closed, negatively-curved Riemannian  $p$ -manifold, the following strengthening of Theorem 1.8 holds by work of Barbot [Ba]:  $\Gamma$  is  $\mathbb{H}^{p,1}$ -convex cocompact if and only if the limit set  $\Lambda_\Gamma$  (topologically a  $(p-1)$ -sphere) is negative, if and only if  $\Lambda_\Gamma$  is *nonpositive* (i.e. it lifts to a cone of  $\mathbb{R}^{p,2} \setminus \{0\}$  on which  $\langle \cdot, \cdot \rangle_{p,2}$  is nonpositive); this property is called *GH-regularity* [Ba, § 1.3]. Using this, Barbot shows that the space of  $P_1^{p,2}$ -Anosov representations of  $\Gamma$  into  $G = \mathrm{PO}(p, 2)$  is not only open but also closed in  $\mathrm{Hom}(\Gamma, G)$ , hence it is a union of connected components of  $\mathrm{Hom}(\Gamma, G)$  [Ba, Th. 1.2]. This becomes false when  $\Gamma$  has virtual cohomological dimension  $< p$ . For instance, when  $\Gamma$  is a finitely generated free group the space  $\mathrm{Hom}(\Gamma, \mathrm{PO}(p, q)_0)$  is connected but contains both Anosov and non-Anosov elements.

**Remark 1.11.** For  $\mathrm{rank}_{\mathbb{R}}(G) := \min(p, q) \geq 2$ , there are examples of irreducible  $P_1^{p,q}$ -Anosov representations  $\rho : \Gamma \rightarrow G = \mathrm{PO}(p, q)$  such that the limit set  $\Lambda_{\rho(\Gamma)} \subset \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  is neither negative nor positive: see Section 5.2. By Theorem 1.8 the group  $\rho(\Gamma)$  is neither  $\mathbb{H}^{p,q-1}$ -convex cocompact nor  $\mathbb{H}^{q,p-1}$ -convex cocompact in this case. In such examples  $\partial_\infty \Gamma$  is always disconnected. Thus one cannot remove the connectedness assumption in Theorem 1.4.(2). This subtlety should be kept in mind while reading [BM, § 8.2].

**Remark 1.12.** Theorems 1.4 and 1.8 and Corollary 1.9 hold even when  $\Gamma$  is not irreducible, for an appropriate, slightly stronger definition of  $\mathbb{H}^{p,q-1}$ -convex cocompactness: see [DGK2].

**1.5. Link with strong projective convex cocompactness.** A properly convex open domain  $\Omega$  in  $\mathbb{P}(\mathbb{R}^n)$  is said to be *strictly convex* if its boundary does not contain any nontrivial segment. It is said to have  *$C^1$  boundary* if every point of the boundary of  $\Omega$  has a unique supporting hyperplane. In [CM], Crampon–Marquis introduced a notion of geometrically finite subgroup  $\Gamma$  of  $\mathrm{PGL}(\mathbb{R}^n)$ , requiring  $\Gamma$  to preserve and act with various nice properties on a strictly convex domain of  $\mathbb{P}(\mathbb{R}^n)$  with  $C^1$  boundary. If cusps are not allowed,



the notion reduces to a natural notion of convex cocompactness. We will refer to this notion as *strong projective convex cocompactness* to distinguish it from Definition 1.2 and from a weaker notion which we study in [DGK2].

**Definition 1.13** ([CM]). A discrete subgroup  $\Gamma$  of  $\mathrm{PGL}(\mathbb{R}^n)$  is *strongly projectively convex cocompact* if it acts properly discontinuously on some nonempty, strictly convex, open subset  $\Omega$  of  $\mathbb{P}(\mathbb{R}^{p,q})$  with  $C^1$  boundary and if the convex hull of  $\Lambda_\Gamma$  in  $\Omega$  has compact quotient by  $\Gamma$ .

We make the following link between Definitions 1.2 and 1.13.

**Proposition 1.14.** *For  $p, q \in \mathbb{N}^*$  with  $p + q \geq 3$ , let  $\Gamma$  be an irreducible discrete subgroup of  $G = \mathrm{PO}(p, q)$ .*

- (1) *If  $\Gamma$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact, then it is strongly projectively convex cocompact. Moreover, the set  $\Omega$  of Definition 1.13 may be taken to be contained in  $\mathbb{H}^{p,q-1}$ .*
- (2) *Conversely, if  $\Gamma$  is strongly projectively convex cocompact, then it is  $\mathbb{H}^{p,q-1}$ -convex cocompact or  $\mathbb{H}^{q,p-1}$ -convex cocompact (after identifying  $\mathrm{PO}(p, q)$  with  $\mathrm{PO}(q, p)$ ).*

In the setting of Proposition 1.14, the compact set  $\Gamma \backslash \mathrm{Conv}(\Lambda_\Gamma)$  is the *convex core* of the real projective manifold  $\Gamma \backslash \Omega$  (or orbifold if torsion is allowed). These convex cocompact real projective manifolds  $\Gamma \backslash \Omega$  provide a natural generalization of the compact real projective manifolds which have been described and classified by Goldman [Go] in dimension 2 and by Benoist [B3, B4, B5, B6] in higher dimension.

The following observation is an easy consequence of the definitions. We refer to [GW] for the notion of  $P_1$ -Anosov representation into  $\mathrm{PGL}(\mathbb{R}^n)$ , sometimes also known as *projective Anosov representation*.

**Fact 1.15** ([GW, Th. 4.3]). *Let  $p, q \in \mathbb{N}^*$  with  $p + q = n$ . A representation with values in  $\mathrm{PO}(p, q)$  is  $P_1^{p,q}$ -Anosov if and only if it is  $P_1$ -Anosov as a representation into  $\mathrm{PGL}(\mathbb{R}^n)$ , where  $P_1$  is the stabilizer of a line of  $\mathbb{R}^n$ .*

Therefore, Theorems 1.4 and 1.8 and Proposition 1.14 give an intimate relationship between  $P_1$ -Anosov representations into  $\mathrm{PGL}(\mathbb{R}^n)$  and strongly projectively convex cocompact subgroups of  $\mathrm{PGL}(\mathbb{R}^n)$  in the context where there is an invariant quadratic form on  $\mathbb{R}^n$ . In [DGK2], we shall generalize this relationship to the setting of subgroups of  $\mathrm{PGL}(\mathbb{R}^n)$  which do not necessarily preserve any quadratic form. The arguments in the proofs of Theorems 1.4 and 1.8 take place in projective geometry, and with some work the use of the quadratic form can be removed, as will be done in [DGK2].

**1.6. Examples of  $\mathbb{H}^{p,q-1}$ -convex cocompact subgroups coming from Anosov representations.** Theorems 1.4 and 1.8 imply that many well-known examples of Anosov representations yield  $\mathbb{H}^{p,q-1}$ -convex cocompact groups. In Section 7, we describe examples, generalizing quasi-Fuchsian representations, that come from deformations of a convex cocompact subgroup

of a rank-one Lie subgroup  $H$  of  $G$ . These include certain maximal representations of surface groups and Hitchin representations. Applying Proposition 1.14, all of these examples are new examples of strongly projectively convex cocompact subgroups of projective linear groups.

Here is a sample result from Section 7.2. We say that a subgroup of  $\mathrm{SO}(p, q)$  is  $\mathbb{H}^{p, q-1}$ -convex cocompact if its image in  $\mathrm{PO}(p, q) = \mathrm{O}(p, q)/\{\pm I\}$  is.

**Proposition 1.16.** *Let  $\Gamma$  be the fundamental group of a closed orientable hyperbolic surface, let  $m \geq 1$ , and let  $\ell \in \{m, m+1\}$ .*

*If  $m$  is odd (resp. even), then the group  $\rho(\Gamma)$  is  $\mathbb{H}^{m+1, \ell-1}$ -convex cocompact (resp.  $\mathbb{H}^{\ell, m}$ -convex cocompact) whenever the irreducible representation  $\rho : \Gamma \rightarrow \mathrm{SO}(m+1, \ell)$  is in the Hitchin component of  $\mathrm{Hom}(\Gamma, \mathrm{SO}(m+1, \ell))$ .*

**1.7. New examples of Anosov representations.** Conversely, Theorem 1.4 also enables us to give new examples of Anosov representations into semisimple Lie groups. While Anosov representations of free groups and surface groups are abundant in the literature, examples of Anosov representations of groups with higher complexity, for instance groups with higher virtual cohomological dimension, are much more rare. We show that certain natural and explicit representations of hyperbolic right-angled Coxeter groups, namely deformations of the Tits canonical representation studied by Krammer [Kr] and others (see e.g. Dyer–Hohlweg–Ripoll [DHR]), are  $\mathbb{H}^{p, q-1}$ -convex cocompact for some appropriate pair  $(p, q)$ ; therefore, by Theorem 1.4, they are  $P_1^{p, q}$ -Anosov.

**Theorem 1.17.** *Let  $W$  be a word hyperbolic right-angled Coxeter group in  $n$  generators. Then  $W$  admits a  $P_1^{p, q}$ -Anosov representation into  $\mathrm{PO}(p, q)$  for some  $p, q \in \mathbb{N}^*$  with  $p + q = n$ . Composing with the inclusion  $\mathrm{PO}(p, q) \hookrightarrow \mathrm{PGL}(\mathbb{R}^n)$  gives a  $P_1$ -Anosov representation of  $W$  into  $\mathrm{PGL}(\mathbb{R}^n)$  (Fact 1.15).*

The class of hyperbolic right-angled Coxeter groups includes, for instance, groups of arbitrarily large virtual cohomological dimension [JS, O]. Theorem 1.17 also provides (by restriction to a subgroup or induction to a finite-index overgroup) Anosov representations for all groups commensurable to hyperbolic right-angled Coxeter groups, as well as Anosov representations for all their quasi-isometrically embedded subgroups: see [DGK2].

**1.8. Organization of the paper.** In Section 2 we recall some well-known facts about the space  $\mathbb{H}^{p, q-1}$  and properly convex domains in projective space. In Section 3 we give a characterization of negative subsets of  $\partial_{\mathbb{P}} \mathbb{H}^{p, q-1}$ , from which we deduce Proposition 1.7 and Corollary 1.9, and we establish some general properties of properly convex domains of  $\mathbb{P}(\mathbb{R}^{p, q})$  preserved by discrete subgroups of  $\mathrm{PO}(p, q)$ . Sections 4 and 5 are devoted to the proofs of implications (i)  $\Rightarrow$  (ii) and (ii)  $\Rightarrow$  (i) of Theorem 1.8, respectively. In Section 6 we prove Proposition 1.14, which makes the link between our notion of  $\mathbb{H}^{p, q-1}$ -convex cocompactness and strong projective convex cocompactness (Definition 1.13). In Section 7 we give examples of  $\mathbb{H}^{p, q-1}$ -convex cocompact



representations coming from well-known families of Anosov representations. Finally, in Section 8 we give the construction of  $\mathbb{H}^{p,q-1}$ -convex cocompact right-angled Coxeter groups and prove Theorem 1.17. In Appendix A we provide a proof of a (probably well known) basic result in point-set topology.

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## 2. REMINDERS AND BASIC FACTS

**2.1. Pseudo-Riemannian hyperbolic spaces.** Fix  $p, q \in \mathbb{N}^*$  with  $p+q \geq 3$ . Let  $G = \mathrm{PO}(p, q)$  and let  $P_1^{p,q}$  be the stabilizer in  $G$  of an isotropic line of  $\mathbb{R}^{p,q}$ . The projective space  $\mathbb{P}(\mathbb{R}^{p+q})$  is the disjoint union of

$$\mathbb{H}^{p,q-1} = \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x, x \rangle_{p,q} < 0\},$$

of

$$\mathbb{S}^{p-1,q} = \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x, x \rangle_{p,q} > 0\},$$

and of

$$\partial_{\mathbb{P}} \mathbb{H}^{p,q-1} = \partial_{\mathbb{P}} \mathbb{S}^{p-1,q} = \{[x] \in \mathbb{P}(\mathbb{R}^{p,q}) \mid \langle x, x \rangle_{p,q} = 0\} \simeq G/P_1^{p,q}.$$

For instance, Figure 1 shows

$$\mathbb{P}(\mathbb{R}^4) = \mathbb{H}^{3,0} \sqcup (\partial_{\mathbb{P}} \mathbb{H}^{3,0} = \partial_{\mathbb{P}} \mathbb{S}^{2,1}) \sqcup \mathbb{S}^{2,1}$$

and

$$\mathbb{P}(\mathbb{R}^4) = \mathbb{H}^{2,1} \sqcup (\partial_{\mathbb{P}} \mathbb{H}^{2,1} = \partial_{\mathbb{P}} \mathbb{S}^{1,2}) \sqcup \mathbb{S}^{1,2}.$$

The space  $\mathbb{H}^{p,q-1}$  is homeomorphic to  $\mathbb{R}^p \times \mathbb{P}(\mathbb{R}^q)$ . It has a natural pseudo-

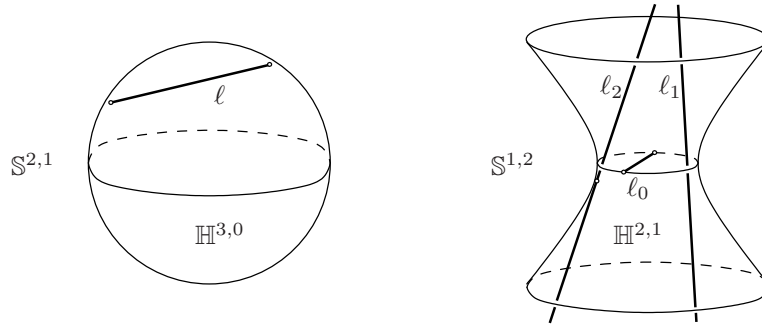


FIGURE 1. Left:  $\mathbb{H}^{3,0}$  with a geodesic line  $\ell$  (necessarily spacelike), and  $\mathbb{S}^{2,1}$ . Right:  $\mathbb{H}^{2,1}$  with three geodesic lines  $\ell_0$  (spacelike),  $\ell_1$  (timelike), and  $\ell_2$  (lightlike), and  $\mathbb{S}^{1,2}$ .

Riemannian structure of signature  $(p, q - 1)$  with isometry group  $G$ . To see this, consider the double covering

$$\widehat{\mathbb{H}}^{p,q-1} = \{x \in \mathbb{R}^{p,q} \mid \langle x, x \rangle_{p,q} = -1\}.$$

The restriction of  $\langle \cdot, \cdot \rangle_{p,q}$  to any tangent space to  $\widehat{\mathbb{H}}^{p,q-1}$  in  $\mathbb{R}^{p,q}$  has signature  $(p, q - 1)$  and defines a pseudo-Riemannian structure on  $\widehat{\mathbb{H}}^{p,q-1}$  with isometry group  $O(p, q)$ , descending to a pseudo-Riemannian structure on  $\mathbb{H}^{p,q-1}$  with isometry group  $PO(p, q)$ . The sectional curvature is constant negative for this pseudo-Riemannian structure. The geodesic lines of the pseudo-Riemannian space  $\mathbb{H}^{p,q-1}$  are the intersections of  $\mathbb{H}^{p,q-1}$  with projective lines in  $\mathbb{P}(\mathbb{R}^{p,q})$ . Such a line is called *spacelike* (resp. *lightlike*, resp. *timelike*) if it meets  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  in two (resp. one, resp. zero) points.

Similarly,  $\mathbb{S}^{p-1,q}$  is homeomorphic to  $\mathbb{P}(\mathbb{R}^p) \times \mathbb{R}^q$  and has a natural pseudo-Riemannian structure of signature  $(p - 1, q)$  with isometry group  $G$ , of constant positive curvature. It identifies with  $\mathbb{H}^{q,p-1}$  as in Remark 1.3.

**Remark 2.1.** For  $(p, q) \neq (2, 2)$ , the group  $G = PO(p, q)$  is simple and  $P_1^{p,q}$  is a maximal proper parabolic subgroup of  $G$ . This is not true for  $(p, q) = (2, 2)$ : the group  $PO(2, 2)_0$  is isomorphic to  $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$  and its subgroup  $P_1^{2,2} \cap PO(2, 2)_0$  is the intersection of two maximal proper parabolic subgroups, namely  $B \times PSL_2(\mathbb{R})$  and  $PSL_2(\mathbb{R}) \times B$  where  $B$  is a Borel subgroup of  $PSL_2(\mathbb{R})$ . To see this, observe that the space  $M_2(\mathbb{R})$  of  $(2 \times 2)$  real matrices is endowed with a natural nondegenerate quadratic form of signature  $(2, 2)$ , namely the determinant; it thus identifies with  $\mathbb{R}^{2,2}$ . The group  $PO(2, 2)_0$  acting on  $\mathbb{R}^{2,2}$  identifies with the group  $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$  acting on  $\mathbb{P}(M_2(\mathbb{R}))$  by right and left multiplication.

The following notation, used in the introduction, will remain valid throughout the paper.

**Notation 2.2.** Let  $X$  be a subset of  $\mathbb{P}(\mathbb{R}^{p,q})$  (e.g. a subset of  $\mathbb{H}^{p,q-1}$ ). We denote by

- $\overline{X}$  the closure of  $X$  in  $\mathbb{P}(\mathbb{R}^{p,q})$ ;
- $\text{Int}(X)$  the interior of  $X$  in  $\mathbb{P}(\mathbb{R}^{p,q})$  (or equivalently in  $\mathbb{H}^{p,q-1}$  if  $X \subset \mathbb{H}^{p,q-1}$ ).

We set  $\partial_{\mathbb{P}}X := \overline{X} \setminus \text{Int}(X)$  and, when  $X \subset \mathbb{H}^{p,q-1}$ , we denote by

- $\partial_{\mathbb{H}}X := \partial_{\mathbb{P}}X \cap \mathbb{H}^{p,q-1}$  the boundary of  $X$  in  $\mathbb{H}^{p,q-1}$ ;
- $\partial_i X := \partial_{\mathbb{P}}X \cap \partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  the boundary of  $X$  in  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ ; if  $X$  is closed in  $\mathbb{H}^{p,q-1}$ , this coincides with the ideal boundary of  $X$ , namely  $\overline{X} \setminus X$ .

We also denote by  $\partial_{\infty}\Gamma$  the Gromov boundary of a word hyperbolic group  $\Gamma$ .

**2.2. Limit sets in projective space.** Let  $V$  be a finite-dimensional real vector space. Recall that an element  $g \in \text{PGL}(V)$  is said to be *proximal* in  $\mathbb{P}(V)$  if it admits a unique attracting fixed point in  $\mathbb{P}(V)$ . Equivalently,  $g$  has a unique complex eigenvalue of maximal modulus. We shall use the following terminology from [Gu, B1].

**Definition 2.3.** Let  $\Gamma$  be an irreducible subgroup of  $\mathrm{PGL}(V)$  containing at least one element which is proximal in  $\mathbb{P}(V)$ . The *limit set* of  $\Gamma$  in  $\mathbb{P}(V)$  is the closure  $\Lambda_\Gamma$  of the set of attracting fixed points of proximal elements of  $\Gamma$ .

By [B1], the action of  $\Gamma$  on this closed,  $\Gamma$ -invariant subset of  $\mathbb{P}(V)$  is *minimal*, i.e. all  $\Gamma$ -orbits are dense; any nonempty, closed,  $\Gamma$ -invariant subset of  $\mathbb{P}(V)$  contains the limit set.

**Remark 2.4.** Suppose  $\Gamma$  is contained in  $G = \mathrm{PO}(p, q)$  for some  $p, q \in \mathbb{N}^*$  with  $p + q = n \geq 3$ . An element  $g \in G$  is proximal in  $\mathrm{PGL}(\mathbb{R}^{p,q})$  if and only if it is proximal in  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ , in the sense that  $g$  admits a unique attracting fixed point in  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ . In this case,  $g^{-1}$  is automatically proximal as well. The limit set of  $\Gamma$  in  $\mathbb{P}(\mathbb{R}^n)$  (Definition 2.3) is contained in  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ , and called the limit set of  $\Gamma$  in  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ .

**2.3. Properly convex domains in projective space.** Let  $\Omega$  be a properly convex open domain in  $\mathbb{P}(V)$ , with boundary  $\partial_{\mathbb{P}}\Omega$ . Recall the *Hilbert metric*  $d_\Omega$  on  $\Omega$ :

$$d_\Omega(y, z) := \frac{1}{2} \log [a, y, z, b]$$

for all distinct  $y, z \in \Omega$ , where  $a, b$  are the intersection points of  $\partial_{\mathbb{P}}\Omega$  with the projective line through  $y$  and  $z$ , with  $a, y, z, b$  in this order. Here  $[\cdot, \cdot, \cdot, \cdot]$  denotes the cross-ratio on  $\mathbb{P}^1(\mathbb{R})$ , normalized so that  $[0, 1, z, \infty] = z$ . The metric space  $(\Omega, d_\Omega)$  is complete and proper, and the automorphism group

$$\mathrm{Aut}(\Omega) := \{g \in \mathrm{PGL}(V) \mid g \cdot \Omega = \Omega\}$$

acts on  $\Omega$  by isometries for  $d_\Omega$ . As a consequence, any discrete subgroup of  $\mathrm{Aut}(\Omega)$  acts properly discontinuously on  $\Omega$ .

Let  $V^*$  be the dual vector space of  $V$ . By definition, the *dual convex set* of  $\Omega$  is

$$\Omega^* := \mathbb{P}(\{\ell \in V^* \mid \ell(x) < 0 \quad \forall x \in \overline{\Omega}\}),$$

where  $\overline{\Omega}$  is the closure in  $V \setminus \{0\}$  of an open convex cone  $\tilde{\Omega}$  of  $V \setminus \{0\}$  lifting  $\Omega$ . The set  $\Omega^*$  is a properly convex open domain in  $\mathbb{P}(V^*)$  which is preserved by the dual action of  $\mathrm{Aut}(\Omega)$  on  $\mathbb{P}(V^*)$ .

Straight lines (contained in projective lines) are always geodesics for the Hilbert metric  $d_\Omega$ . When  $\Omega$  is not strictly convex, there can be other geodesics as well, by the following well-known and easy fact.

**Fact 2.5.** *For pairwise distinct points  $w_1, w_2, w_3 \in \Omega$ , we have  $d_\Omega(w_1, w_3) = d_\Omega(w_1, w_2) + d_\Omega(w_2, w_3)$  if and only if there are segments  $[y, y']$  and  $[z, z']$  in the boundary of  $\Omega$  such that  $y, w_1, w_2, z$  on the one hand, and  $y', w_2, w_3, z'$  on the other hand, are aligned in this order.*

However, the following fact is always true, and will be used in Section 4.4. It was proved by Foertsch–Karlsson [FK, Th.3]; here we provide a short proof for the reader's convenience.

**Lemma 2.6.** *Any biinfinite geodesic of  $(\Omega, d_\Omega)$  has well-defined, distinct endpoints in the boundary  $\partial_{\mathbb{P}}\Omega$ .*

*Proof.* We work in an affine Euclidean chart where  $\Omega$  is bounded. Let  $\mathcal{G} = (\mathcal{G}(t))_{t \in \mathbb{R}}$  be a biinfinite geodesic of  $(\Omega, d_\Omega)$ . For any  $s < t$  in  $\mathbb{R}$ , let  $y_{s,t} \in \partial_{\mathbb{P}}\Omega$  and  $z_{s,t} \in \partial_{\mathbb{P}}\Omega$  be such that  $y_{s,t}, \mathcal{G}(s), \mathcal{G}(t), z_{s,t}$  are aligned in this order.

We claim that all points  $y_{s,t}$  for  $s < t$  are contained in a common face  $P$  of  $\partial_{\mathbb{P}}\Omega$  (of arbitrary dimension). Indeed, for any  $s < t$ , let  $P_{s,t}$  be the intersection of  $\partial_{\mathbb{P}}\Omega$  and of all supporting hyperplanes to  $\Omega$  at  $y_{s,t}$ ; it is a nonempty compact subset of  $\partial_{\mathbb{P}}\Omega$ . We claim that  $P_{u,v} \subset P_{s,t}$  for  $s < u < v < t$ . Indeed, by Fact 2.5 we have  $y_{s,t} \in [y_{s,u}, y_{u,t}]$  (see Figure 2, left), hence any supporting hyperplane to  $\Omega$  at  $y_{s,t}$  is also a supporting hyperplane to  $\Omega$  at  $y_{s,u}$  and  $y_{u,t}$ ; applying the same reasoning to  $(u, v, t)$  instead of  $(s, u, t)$ , we see that any supporting hyperplane to  $\Omega$  at  $y_{s,t}$  is also a supporting hyperplane to  $\Omega$  at  $y_{u,v}$ , proving  $P_{u,v} \subset P_{s,t}$ . Thus we have a sequence  $(P_{-n,n})_{n \geq 1}$  of nonempty compact subsets of  $\partial_{\mathbb{P}}\Omega$  which is nondecreasing for inclusion; it must have a limit  $P$ .

Similarly, all points  $z_{s,t}$  for  $s < t$  are contained in a common face  $Q$  of  $\partial_{\mathbb{P}}\Omega$ .

Any forward accumulation point  $a$  of  $\mathcal{G}$  in the boundary of  $\Omega$  is an accumulation point of the  $z_{0,t}$  as  $t \rightarrow +\infty$ , and therefore belongs to  $Q$ . In the Euclidean metric,  $z_{0,t}$  for  $t > 0$  is further away from the span of  $P$  than  $\mathcal{G}(0)$  is: therefore  $a$  belongs in fact to  $Q \setminus P$ . (In particular  $P \neq Q$ .) Similarly, all backward accumulation points of  $\mathcal{G}$  are in  $P \setminus Q$ .

Suppose by contradiction that there are two sequences  $(s_n)_{n \in \mathbb{N}}$  and  $(t_n)_{n \in \mathbb{N}}$  of positive numbers tending to  $+\infty$  such that  $\mathcal{G}(s_n)$  and  $\mathcal{G}(t_n)$  converge respectively to some points  $a \neq b$  in  $Q \setminus P$ . Up to taking subsequences, we may assume that  $s_n < t_n < s_{n+1}$  for all  $n$ .

Consider the triangle  $T_n$  spanned by  $y_{s_n, t_n}$ ,  $\mathcal{G}(t_n)$ , and  $y_{t_n, s_{n+1}}$  (see Figure 2, right). Its angle at  $\mathcal{G}(t_n)$  goes to  $\pi$  in the chosen chart as  $n \rightarrow +\infty$ , because  $\mathcal{G}(s_n), \mathcal{G}(s_{n+1}) \rightarrow a$  and  $t_n \rightarrow b$ . The opposite edge  $[y_{s_n, t_n}, y_{t_n, s_{n+1}}]$  of  $T_n$  converges to a segment of  $P$  as  $n \rightarrow +\infty$ : therefore  $\lim_n \mathcal{G}(t_n) \in P$ . But  $\mathcal{G}(t_n) \rightarrow b \notin P$ : contradiction. Thus  $\mathcal{G}$  has a unique forward endpoint  $a$  in the boundary of  $\Omega$ , belonging to  $Q \setminus P$ . Similarly, it has a unique backward endpoint  $a' \neq a$  in  $P \setminus Q$ .  $\square$

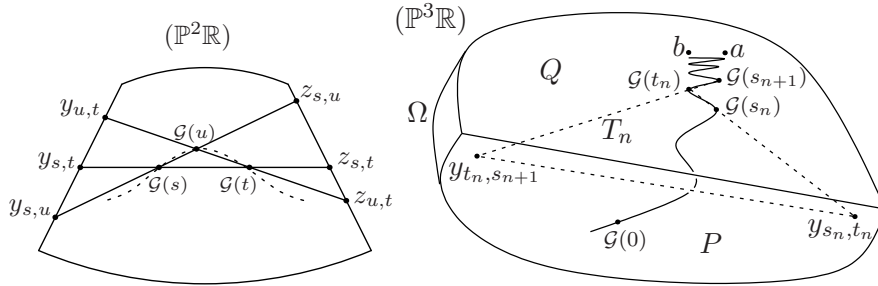


FIGURE 2. Illustrations for the proof of Lemma 2.6. Left: Definition of the points  $y_{s,t}$  and  $z_{s,t}$ . Right: Absurd situation where  $\mathcal{G}$  would have two forward accumulation points  $a \neq b$ .

In Section 6.1 we shall also use the following elementary observation.

**Lemma 2.7.** *Let  $\mathcal{O}$  be a properly convex open domain of  $\mathbb{P}(V)$  and let  $\mathcal{O}_1, \dots, \mathcal{O}_\ell$  be convex open subsets with convex hull  $\mathcal{O}$ . If the boundary  $\partial_{\mathbb{P}} \mathcal{O}_i$  is  $C^1$  for every  $i$ , then so is the boundary  $\partial_{\mathbb{P}} \mathcal{O}$ .*

*Proof.* For any  $i$ , the boundary  $\partial_{\mathbb{P}} \mathcal{O}_i$  is  $C^1$  if and only if the dual convex set  $\mathcal{O}_i^*$  is strictly convex. In this case, the intersection  $\bigcap_{i=1}^j \mathcal{O}_i^*$  is also strictly convex. But this intersection is the dual  $\mathcal{O}^*$  of  $\mathcal{O}$ . Therefore,  $\partial_{\mathbb{P}} \mathcal{O}$  is  $C^1$ .  $\square$

**2.4. Irreducible groups preserving properly convex domains.** We shall use the following general properties due to Benoist [B2, Prop. 3.1].

**Fact 2.8** ([B2]). *An irreducible subgroup  $\Gamma$  of  $\mathrm{PGL}(V)$  preserves a nonempty properly convex subset of  $\mathbb{P}(V)$  if and only if the following two conditions are simultaneously satisfied:*

- (i)  $\Gamma$  contains an element of  $\mathrm{PGL}(V)$  which is proximal both in  $\mathbb{P}(V)$  and in  $\mathbb{P}(V^*)$ ,
- (ii)  $\Lambda_\Gamma$  and  $\Lambda_\Gamma^*$  lift respectively to cones  $\tilde{\Lambda}_\Gamma$  of  $V \setminus \{0\}$  and  $\tilde{\Lambda}_\Gamma^*$  of  $V^* \setminus \{0\}$  such that  $\ell(x) \leq 0$  for all  $x \in \tilde{\Lambda}_\Gamma$  and  $\ell \in \tilde{\Lambda}_\Gamma^*$ .

In this case,

- (1) for any  $\Gamma$ -invariant convex open domain  $\Omega$ , the limit set  $\Lambda_\Gamma$  (resp.  $\Lambda_\Gamma^*$ ) is contained in the boundary of  $\Omega$  (resp.  $\Omega^*$ ) in  $\mathbb{P}(V)$  (resp.  $\mathbb{P}(V^*)$ );
- (2) there is a smallest nonempty,  $\Gamma$ -invariant, convex open domain  $\Omega_{\min}$  of  $\mathbb{P}(V)$ , namely the projectivization of the interior of the  $\mathbb{R}^+$ -span of  $\tilde{\Lambda}_\Gamma$ ; it is contained in any  $\Gamma$ -invariant convex open domain  $\Omega$  of  $\mathbb{P}(V)$ , and is the interior of the convex hull of  $\Lambda_\Gamma$  in  $\Omega$ ;
- (3) there is a largest  $\Gamma$ -invariant convex open domain  $\Omega_{\max}$  of  $\mathbb{P}(V)$ , namely the dual convex set to the projectivization of the interior of the  $\mathbb{R}^+$ -span of  $\tilde{\Lambda}_\Gamma^*$ ; it contains any  $\Gamma$ -invariant convex open domain of  $\mathbb{P}(V)$  and is properly convex.

**Remark 2.9.** When  $V = \mathbb{R}^{p,q}$  and  $\Gamma$  is contained in  $\mathrm{PO}(p, q)$ , the map  $x \mapsto \langle x, \cdot \rangle_{p,q}$  from  $V$  to  $V^*$  induces a homeomorphism  $\Lambda_\Gamma \simeq \Lambda_\Gamma^*$ , and  $\Omega_{\max}$  is a connected component of the complement, in  $V$ , of the union of the projective hyperplanes  $z^\perp$  for  $z \in \Lambda_\Gamma$ . In fact  $\Omega_{\max}$  is the unique such connected component containing  $\Lambda_\Gamma$  in its boundary. In the Lorentzian case (i.e.  $q = 2$ ), the set  $\Omega_{\max}$  is called the *invisible domain* of  $\Lambda_\Gamma$ .

### 3. NONPOSITIVE SUBSETS OF $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$

We shall use the following terminology which extends Definition 1.6.

**Definition 3.1.** A subset  $\Lambda$  of  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  is *negative* (resp. *nonpositive*, resp. *nonnegative*, *positive*) if it lifts to a cone of  $\mathbb{R}^{p,q} \setminus \{0\}$  on which all inner products  $\langle \cdot, \cdot \rangle_{p,q}$  of noncolinear points are negative (resp. nonpositive, resp. nonnegative, resp. positive).

In the Lorentzian case ( $q = 2$ ), the usual terminology for a negative (resp. nonpositive) subset of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  is *acausal* (resp. *achronal*).

**3.1. Reading the sign on triples.** The following characterization will be used only to prove Proposition 1.7 and Corollary 1.9 (in Section 3.2 below).

**Lemma 3.2.** *Let  $\Lambda$  be a subset of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  with at least three points. Then the following are equivalent:*

- (i)  $\Lambda$  is negative,
- (ii) every triple of distinct points of  $\Lambda$  is negative,
- (iii) every triple of distinct points of  $\Lambda$  spans a triangle fully contained in  $\mathbb{H}^{p,q-1}$  outside of the vertices.

The equivalence (ii)  $\Leftrightarrow$  (iii) is contained in the following immediate remark.

**Remark 3.3.** For any pairwise distinct points  $y_1, y_2, y_3$  of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ , the following are equivalent:

- there exist lifts  $x_i \in \mathbb{R}^{p,q} \setminus \{0\}$  of  $y_i$  such that  $\langle x_i, x_j \rangle_{p,q} < 0$  for all  $i \neq j$ ,
- for any lifts  $x_i \in \mathbb{R}^{p,q} \setminus \{0\}$  of  $y_i$ , we have

$$\langle x_1, x_2 \rangle_{p,q} \langle x_1, x_3 \rangle_{p,q} \langle x_2, x_3 \rangle_{p,q} < 0,$$

- there exist lifts  $x_i \in \mathbb{R}^{p,q} \setminus \{0\}$  of  $y_i$  such that for any  $t_i \geq 0$ , if at least two of the  $t_i$  are nonzero, then

$$\left\langle \sum_{i=1}^3 t_i x_i, \sum_{i=1}^3 t_i x_i \right\rangle_{p,q} = \sum_{1 \leq i < j \leq 3} 2t_i t_j \langle x_i, x_j \rangle_{p,q} < 0,$$

- $(y_1, y_2, y_3)$  spans a triangle fully contained in  $\mathbb{H}^{p,q-1}$  outside of the vertices.

*Proof of Lemma 3.2.* If  $\Lambda$  is negative, then any subset of  $\Lambda$  is as well, and so (i)  $\Rightarrow$  (ii) holds. We now check (ii)  $\Rightarrow$  (i).

Suppose that every triple of distinct points of  $\Lambda$  is negative. Choose two distinct points  $y_1, y_2 \in \Lambda$  and respective lifts  $x_1, x_2 \in \mathbb{R}^{p,q} \setminus \{0\}$  with  $\langle x_1, x_2 \rangle_{p,q} < 0$ . We now define a map  $f : \Lambda \rightarrow \mathbb{R}^{p,q} \setminus \{0\}$  as follows. We set  $f(y_i) := x_i$  for  $i \in \{1, 2\}$ . For each  $y \in \Lambda \setminus \{y_1, y_2\}$ , we choose a lift  $x \in \mathbb{R}^{p,q} \setminus \{0\}$  of  $y$ ; by Remark 3.3, we have  $\langle x_1, x_2 \rangle_{p,q} \langle x_1, x \rangle_{p,q} \langle x_2, x \rangle_{p,q} < 0$ , and so  $\langle x_1, x \rangle_{p,q}$  and  $\langle x_2, x \rangle_{p,q}$  are both nonzero of the same sign; we set  $f(y) := x$  if this sign is negative, and  $f(y) := -x$  otherwise. We claim that  $\langle f(y), f(y') \rangle_{p,q} < 0$  for any  $y \neq y'$  in  $\Lambda$ . Indeed, this is true by construction if  $y$  or  $y'$  is equal to  $y_1$ , so we assume this is not the case. By Remark 3.3, we have  $\langle x_1, f(y) \rangle_{p,q} \langle x_1, f(y') \rangle_{p,q} \langle f(y), f(y') \rangle_{p,q} < 0$ . Since  $\langle x_1, f(y) \rangle_{p,q} < 0$  and  $\langle x_1, f(y') \rangle_{p,q} < 0$  by construction, we have  $\langle f(y), f(y') \rangle_{p,q} < 0$ . Thus  $\{tf(y) \mid y \in \Lambda\}$  is a cone of  $\mathbb{R}^{p,q} \setminus \{0\}$  lifting  $\Lambda$  on which all inner products  $\langle \cdot, \cdot \rangle_{p,q}$  of noncolinear points are negative, and so  $\Lambda$  is negative.  $\square$



**Remark 3.4.** Similar equivalences to Lemma 3.2 hold after replacing *negative* with *positive* in conditions (i) and (ii), and  $\mathbb{H}^{p,q-1}$  with  $\mathbb{S}^{p-1,q}$  in condition (iii).

**Remark 3.5.** It follows from Remark 3.3 that a triple of distinct points of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  cannot be both negative and positive. Therefore, by Lemma 3.2, an arbitrary subset of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  with at least three points cannot be both negative and positive.

### 3.2. Consequences of Lemma 3.2.

*Proof of Proposition 1.7.* Let  $\Lambda$  be a closed, connected, transverse subset of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ , and let  $\Lambda^{(3)}$  be the set of unordered triples of distinct points of  $\Lambda$ . By Remark 3.3, we may define a function  $\Lambda^{(3)} \rightarrow \{\pm 1\}$  by sending  $(y_1, y_2, y_3) \in \Lambda^{(3)}$  to the sign of  $\langle x_1, x_2 \rangle_{p,q} \langle x_1, x_3 \rangle_{p,q} \langle x_2, x_3 \rangle_{p,q}$ , where  $x_i \in \mathbb{R}^{p,q} \setminus \{0\}$  is an arbitrary lift of  $y_i$ . This function is continuous and  $\Lambda^{(3)}$  is connected by Fact A.1, hence the function is constant. In other words, every triple of distinct points of  $\Lambda$  is negative or every triple of distinct points of  $\Lambda$  is positive. By Lemma 3.2 and Remark 3.4, the set  $\Lambda$  is negative or positive.  $\square$

Here is another consequence of Lemma 3.2.

**Proposition 3.6.** *Let  $\Gamma$  be a word hyperbolic group and  $\mathcal{T}$  a connected component in the space of  $P_1^{p,q}$ -Anosov representations of  $\Gamma$  into  $G = \mathrm{PO}(p, q)$ . If the limit set  $\Lambda_{\rho(\Gamma)}$  is negative (resp. positive) for some  $\rho \in \mathcal{T}$ , then it is negative (resp. positive) for all  $\rho \in \mathcal{T}$ .*

*Proof.* We may assume  $\#\partial_{\infty}\Gamma \geq 3$ , otherwise for any  $P_1^{p,q}$ -Anosov representation  $\rho : \Gamma \rightarrow G$  the limit set  $\Lambda_{\rho(\Gamma)}$  is both negative and positive.

Suppose  $\Lambda_{\rho_0(\Gamma)}$  is negative for some  $\rho_0 \in \mathcal{T}$ , and let  $(\rho_t)_{t \in [0,1]}$  be a continuous path in  $\mathcal{T}$ . For  $t \in [0, 1]$ , let  $\xi_t : \partial_{\infty}\Gamma \rightarrow \partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  be the boundary map of the Anosov representation  $\rho_t$ . For any triple  $\{\eta_1, \eta_2, \eta_3\}$  of distinct points of  $\partial_{\infty}\Gamma$  and any  $t \in [0, 1]$ , the triple  $\{\xi_t(\eta_1), \xi_t(\eta_2), \xi_t(\eta_3)\} \subset \Lambda_{\rho_t(\Gamma)}$  is either negative or positive, by transversality of  $\xi_t$ . Since  $\{\xi_0(\eta_1), \xi_0(\eta_2), \xi_0(\eta_3)\}$  is negative and  $t \mapsto \xi_t(\eta_i)$  is continuous for all  $i$  (see [GW, Th. 5.13]), we deduce as in the proof of Proposition 1.7 that  $\{\xi_t(\eta_1), \xi_t(\eta_2), \xi_t(\eta_3)\}$  is negative for all  $t \in [0, 1]$ . By Lemma 3.2, the set  $\Lambda_{\rho_t(\Gamma)}$  is negative for all  $t \in [0, 1]$ .

The case that  $\Lambda_{\rho_0(\Gamma)}$  is positive is similar.  $\square$

Corollary 1.9 is a consequence of Theorem 1.8, of the fact [L, GW] that the set of  $P_1^{p,q}$ -Anosov representations is open in  $\mathrm{Hom}(\Gamma, G)$ , and of Proposition 3.6.

**3.3. Boundaries of convex subsets of  $\mathbb{H}^{p,q-1}$ .** The following lemma makes a link between convexity in  $\mathbb{H}^{p,q-1}$  and nonpositivity in  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ .

**Lemma 3.7.** *(1) Let  $\Lambda_0$  be a closed nonpositive (resp. nonnegative) subset of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  which is not contained in a projective hyperplane. Then  $\Lambda_0$*

spans a nonempty convex open domain  $\Omega$  of  $\mathbb{P}(\mathbb{R}^{p,q})$  which is contained in  $\mathbb{H}^{p,q-1}$  (resp. in  $\mathbb{S}^{p-1,q}$ ). Moreover, the intersection  $\Lambda_1 \supset \Lambda_0$  of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  with the closure of  $\Omega$  is still nonpositive (resp. nonnegative).

- (2) Conversely, for any nonempty properly convex open domain  $\Omega$  of  $\mathbb{H}^{p,q-1}$  (resp.  $\mathbb{S}^{p-1,q}$ ), the intersection of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  with the closure of  $\Omega$  is nonpositive (resp. nonnegative).

*Proof.* (1) Let  $\tilde{\Lambda}_0$  be a cone of  $\mathbb{R}^{p,q} \setminus \{0\}$  on which  $\langle \cdot, \cdot \rangle_{p,q}$  is nonpositive (the nonnegative case is similar). Using the equality

$$(3.1) \quad \left\langle \sum_i t_i x_i, \sum_i t_i x_i \right\rangle_{p,q} = \sum_{i,j} 2t_i \langle x_i, x_j \rangle_{p,q}$$

for  $t_i \in \mathbb{R}^+$  and  $x_i \in \tilde{\Lambda}_0$ , we see that  $\langle \cdot, \cdot \rangle_{p,q}$  is still nonpositive on the  $\mathbb{R}^+$ -span of  $\tilde{\Lambda}_0$ . In particular,  $\Lambda_1$  is nonpositive since it is contained in the projectivization of the  $\mathbb{R}^+$ -span of  $\tilde{\Lambda}_0$ . Let  $\Omega$  be the projectivization of the interior of this  $\mathbb{R}^+$ -span. Then  $\Omega$  is convex, contained in  $\mathbb{H}^{p,q-1} \cup \partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  and open, hence contained in  $\mathbb{H}^{p,q-1}$  (since  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  is a hypersurface of  $\mathbb{P}(\mathbb{R}^{p,q})$ ).

(2) Let  $\Omega$  be a nonempty properly convex open domain of  $\mathbb{H}^{p,q-1}$  (the  $\mathbb{S}^{p-1,q}$  case is similar). We can lift it to a properly convex open cone  $\tilde{\Omega}$  of  $\mathbb{R}^{p,q} \setminus \{0\}$ , such that  $\langle x, x \rangle_{p,q} < 0$  for all  $x \in \tilde{\Omega}$ . Let  $\Lambda_1$  be the intersection of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  with the closure of  $\Omega$ , and let  $\tilde{\Lambda}_1$  be a cone of  $\mathbb{R}^{p,q} \setminus \{0\}$  lifting  $\Lambda_1$ , contained in the closure of  $\tilde{\Omega}$ . Using the equality (3.1) for  $t_i \in \mathbb{R}^+$  and  $x_i \in \tilde{\Lambda}_1$ , we see that  $\langle \cdot, \cdot \rangle_{p,q}$  is nonpositive on  $\tilde{\Lambda}_1$ . Thus  $\Lambda_1$  is nonpositive.  $\square$

**3.4. Irreducible subgroups of  $\mathrm{PO}(p, q)$  preserving properly convex domains.** Here is a consequence of Fact 2.8 (see Figure 3).

**Proposition 3.8.** *For  $p, q \in \mathbb{N}^*$  with  $p + q \geq 3$ , an irreducible subgroup  $\Gamma$  of  $G = \mathrm{PO}(p, q)$  preserves a nonempty properly convex subset of  $\mathbb{P}(\mathbb{R}^{p,q})$  if and only if the following two conditions are simultaneously satisfied:*

- (i)  $\Gamma$  contains an element of  $G$  which is proximal in  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ ,
- (ii)  $\Lambda_{\Gamma}$  is nonpositive or nonnegative (Definition 3.1).

*In this case, let  $\tilde{\Lambda}_{\Gamma}$  be a cone of  $\mathbb{R}^{p,q} \setminus \{0\}$  lifting  $\Lambda_{\Gamma}$  on which  $\langle \cdot, \cdot \rangle_{p,q}$  is nonpositive (resp. nonnegative). There is a largest  $\Gamma$ -invariant convex open domain  $\Omega_{\max}$  of  $\mathbb{P}(\mathbb{R}^{p,q})$ , namely the projectivization of the interior of the set of  $x' \in \mathbb{R}^{p,q}$  such that  $\langle x, x' \rangle_{p,q} \leq 0$  for all  $x \in \tilde{\Lambda}_{\Gamma}$  (resp.  $\langle x, x' \rangle_{p,q} \geq 0$  for all  $x \in \tilde{\Lambda}_{\Gamma}$ ); it is properly convex. There is a smallest nonempty  $\Gamma$ -invariant convex open domain of  $\mathbb{P}(\mathbb{R}^{p,q})$ , namely the interior  $\Omega_{\min}$  of the convex hull of  $\Lambda_{\Gamma}$  in  $\Omega_{\max}$ . Any nonempty  $\Gamma$ -invariant convex open domain  $\Omega$  of  $\mathbb{P}(\mathbb{R}^{p,q})$  satisfies  $\Omega_{\min} \subset \Omega \subset \Omega_{\max}$ .*

*Proof.* We only need to check that condition (i) of Proposition 3.8 is equivalent to condition (i) of Fact 2.8, that condition (ii) of Proposition 3.8 implies condition (ii) of Fact 2.8, and that condition (ii) of Fact 2.8 together with the

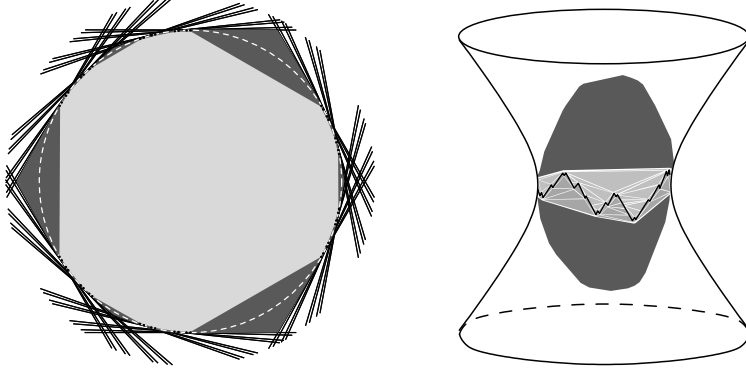


FIGURE 3. The sets  $\Omega_{\min} \subset \mathbb{H}^{p,q-1}$  (light gray) and  $\Omega_{\max} \subset \mathbb{P}(\mathbb{R}^{p,q})$  (dark gray) for a negative limit set  $\Lambda_\Gamma$ . On the left  $(p, q) = (2, 1)$ , and on the right  $(p, q) = (2, 2)$ .

existence of a nonempty  $\Gamma$ -invariant properly convex open domain of  $\mathbb{P}(\mathbb{R}^{p,q})$  implies condition (ii) of Proposition 3.8.

An element  $g \in G = \text{PO}(p, q)$  is proximal in  $\mathbb{P}(\mathbb{R}^{p,q})$  if and only if it is proximal in  $\mathbb{P}((\mathbb{R}^{p,q})^*)$ , because the set of eigenvalues of  $g$  is stable under taking inverses, and so  $g$  has a unique eigenvalue of maximal modulus if and only if  $g^{-1}$  has. The element  $g$  is proximal in  $\mathbb{P}(\mathbb{R}^{p,q})$  if and only if it is proximal in  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ , because all eigenspaces of  $g$  (for eigenvalues of modulus  $\neq 1$ ) project to subsets of  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  (see Section 2.2). Thus condition (i) of Fact 2.8 is equivalent to condition (i) of Proposition 3.8 for  $\Gamma \subset G = \text{PO}(p, q)$ .

Suppose that  $\Lambda_\Gamma$  is nonpositive (resp. nonnegative), i.e. we can lift it to a cone  $\tilde{\Lambda}_\Gamma$  of  $\mathbb{R}^{p,q} \setminus \{0\}$  on which  $\langle \cdot, \cdot \rangle_{p,q}$  is nonpositive (resp. nonnegative). Recall from Remark 2.9 that  $\psi : x \mapsto \langle x, \cdot \rangle_{p,q}$  identifies  $\mathbb{R}^{p,q}$  with  $(\mathbb{R}^{p,q})^*$  and  $\Lambda_\Gamma \subset \mathbb{P}(\mathbb{R}^{p,q})$  with  $\Lambda_\Gamma^* \subset \mathbb{P}((\mathbb{R}^{p,q})^*)$ . Thus the set  $\tilde{\Lambda}_\Gamma^* := \psi(\tilde{\Lambda}_\Gamma)$  (resp.  $\tilde{\Lambda}_\Gamma^* := -\psi(\tilde{\Lambda}_\Gamma)$ ) is a cone of  $(\mathbb{R}^{p,q})^* \setminus \{0\}$  lifting  $\Lambda_\Gamma^*$ , and by construction  $\ell(x) \leq 0$  for all  $x \in \tilde{\Lambda}_\Gamma$  and  $\ell \in \tilde{\Lambda}_\Gamma^*$ . Thus condition (ii) of Proposition 3.8 implies condition (ii) of Fact 2.8 for  $\Gamma \subset G = \text{PO}(p, q)$ .

Suppose that  $\Lambda_\Gamma$  and  $\Lambda_\Gamma^*$  lift respectively to cones  $\tilde{\Lambda}_\Gamma$  of  $\mathbb{R}^{p,q} \setminus \{0\}$  and  $\tilde{\Lambda}_\Gamma^*$  of  $(\mathbb{R}^{p,q})^* \setminus \{0\}$  such that  $\ell(x) \leq 0$  for all  $x \in \tilde{\Lambda}_\Gamma$  and  $\ell \in \tilde{\Lambda}_\Gamma^*$ , and that  $\Gamma$  preserves a nonempty properly convex subset of  $\mathbb{P}(\mathbb{R}^{p,q})$ . Since  $\psi$  induces an identification between  $\Lambda_\Gamma$  and  $\Lambda_\Gamma^*$ , we have  $\psi(x) \in \tilde{\Lambda}_\Gamma^* \cup -\tilde{\Lambda}_\Gamma^*$  for all  $x \in \tilde{\Lambda}_\Gamma$ . Let  $F^-$  (resp.  $F^+$ ) be the subcone of  $\tilde{\Lambda}_\Gamma$  defined by  $\psi(x) \in \tilde{\Lambda}_\Gamma^*$  (resp.  $\psi(x) \in -\tilde{\Lambda}_\Gamma^*$ ). By construction, we have  $x \in F^-$  if and only if  $\langle x, x' \rangle_{p,q} \leq 0$  for all  $x' \in \tilde{\Lambda}_\Gamma$ ; in particular,  $F^-$  is closed in  $\tilde{\Lambda}_\Gamma$ . Similarly,  $x \in F^+$  if and only if  $\langle x, x' \rangle_{p,q} \geq 0$  for all  $x' \in \tilde{\Lambda}_\Gamma$ , and  $F^+$  is closed in  $\tilde{\Lambda}_\Gamma$ . The sets  $F^-$  and  $F^+$  are disjoint since no  $x \in \mathbb{R}^{p,q} \setminus \{0\}$  can satisfy  $\langle x, x' \rangle_{p,q} = 0$  for all  $x' \in \tilde{\Lambda}_\Gamma$ , otherwise the  $\Gamma$ -invariant subset  $\tilde{\Lambda}_\Gamma$  of  $\mathbb{R}^{p,q}$  would be contained in

the hyperplane  $x^\perp$ , contradicting the irreducibility of  $\Gamma$ . Thus  $F^-$  and  $F^+$  are disjoint,  $\Gamma$ -invariant, closed subsets of  $\tilde{\Lambda}_\Gamma$ , whose projections to  $\mathbb{P}(\mathbb{R}^{p,q})$  are disjoint,  $\Gamma$ -invariant, closed subsets of  $\Lambda_\Gamma$ . Since  $\Gamma$  is irreducible,  $\Lambda_\Gamma$  is the smallest nonempty  $\Gamma$ -invariant closed subset of  $\mathbb{P}(\mathbb{R}^{p,q})$  (see Section 2.2), and so  $\tilde{\Lambda}_\Gamma = F^-$  or  $\tilde{\Lambda}_\Gamma = F^+$ . In the first case  $\Lambda_\Gamma$  is nonpositive, and in the second case it is nonnegative.  $\square$

In the setting of Proposition 3.8, if  $\Lambda_\Gamma \subset \partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  is nonpositive (resp. nonnegative), then  $\Omega_{\min}$  is contained in  $\mathbb{H}^{p,q-1}$  (resp.  $\mathbb{S}^{p-1,q}$ ) by Lemma 3.7. We shall use the following in the stronger situation that  $\Lambda_\Gamma$  is negative (resp. positive).

**Lemma 3.9.** *In the setting of Proposition 3.8, if  $\Lambda_\Gamma \subset \partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  is negative (resp. positive), then the closure  $\mathcal{C}_{\min}$  of  $\Omega_{\min}$  in  $\mathbb{H}^{p,q-1}$  (resp.  $\mathbb{S}^{p-1,q}$ ) is contained in  $\Omega_{\max}$ .*

*Proof.* Suppose  $\Lambda_\Gamma$  is negative. For any  $x \in \tilde{\Lambda}_\Gamma$ , using the equality

$$\left\langle x, \sum_i t_i x_i \right\rangle_{p,q} = \sum_i t_i \langle x, x_i \rangle_{p,q}$$

for  $t_i \in \mathbb{R}^+$  and  $x_i \in \tilde{\Lambda}_\Gamma$ , we see that  $\langle x, \cdot \rangle_{p,q}$  is negative on the  $\mathbb{R}^+$ -span of  $\tilde{\Lambda}_\Gamma$  minus  $\{0\}$ . In particular, the set  $\mathcal{C}_{\min}$ , which is the projectivization of this  $\mathbb{R}^+$ -span minus  $\{0\}$ , is contained in  $\Omega_{\max}$ , which is the projectivization of the interior of the set of  $x' \in \mathbb{R}^{p,q}$  such that  $\langle x, x' \rangle_{p,q} \leq 0$  for all  $x \in \tilde{\Lambda}_\Gamma$ .

The case that  $\Lambda_\Gamma$  is positive is analogous.  $\square$

#### 4. $\mathbb{H}^{p,q-1}$ -CONVEX COCOMPACT GROUPS ARE ANOSOV

The goal of this section is to prove the implications (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (iv) of Theorem 1.8, which contain Theorem 1.4.(1). By the following observation, which is immediate from the definitions, we can focus on (i)  $\Rightarrow$  (ii) only.

**Remark 4.1.** A representation  $\rho : \Gamma \rightarrow \mathrm{PO}(p, q)$  is  $P_1^{p,q}$ -Anosov if and only if it is  $P_1^{q,p}$ -Anosov under the identification  $\mathrm{PO}(p, q) \simeq \mathrm{PO}(q, p)$ . A subset of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  is positive if and only if it is negative under the identification  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1} \simeq \partial_{\mathbb{P}}\mathbb{H}^{q,p-1}$ .

Let  $\Gamma$  be an irreducible discrete subgroup of  $G = \mathrm{PO}(p, q)$ . Suppose that condition (i) of Theorem 1.8 holds, namely  $\Gamma$  acts properly discontinuously and cocompactly on some nonempty, properly convex, closed subset  $\mathcal{C}$  of  $\mathbb{H}^{p,q-1}$ . Let  $\Lambda = \partial_i \mathcal{C}$  be the ideal boundary of  $\mathcal{C}$ , i.e. the intersection of  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  with the closure of  $\mathcal{C}$  in  $\mathbb{P}(\mathbb{R}^{p,q})$  (see Notation 2.2). The set  $\Lambda$  contains the limit set  $\Lambda_\Gamma$  by Fact 2.8 and Remark 2.9.

**4.1. Working inside a properly convex open domain.** Let  $\Omega$  be the interior of  $\mathcal{C}$  and  $\Omega_{\max} \supset \Omega$  the largest  $\Gamma$ -invariant convex open domain of  $\mathbb{P}(\mathbb{R}^{p,q})$  (Fact 2.8 and Remark 2.9). Since  $\Gamma$  is irreducible,  $\Omega$  is nonempty and

$\Omega_{\max}$  is properly convex (Fact 2.8). The following lemma, which requires the properness and cocompactness of the action of  $\Gamma$  on  $\mathcal{C}$ , will enable us to later use the restriction to  $\mathcal{C}$  of the Hilbert metric of  $\Omega_{\max}$ .

**Lemma 4.2.** *The set  $\mathcal{C}$ , which is a closed subset of  $\mathbb{H}^{p,q-1}$ , is contained in the open set  $\Omega_{\max}$ .*

*Proof.* Suppose by contradiction that  $\mathcal{C}$  is not contained in  $\Omega_{\max}$ . By Fact 2.8 and Remark 2.9, this means that some point  $y \in \mathcal{C}$  belongs to  $z^\perp$  for some  $z \in \Lambda_\Gamma$ . By convexity of  $\mathcal{C}$ , the interval  $[y, z]$  is a lightlike ray of  $\mathbb{H}^{p,q-1}$  fully contained in  $\mathcal{C}$ . Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of points of  $[y, z]$  converging to  $z$  (see Figure 4). Since  $\Gamma$  acts cocompactly on  $\mathcal{C}$ , for any  $n$  there exists  $\gamma_n \in \Gamma$  such that  $\gamma_n \cdot a_n$  belongs to a fixed compact subset of  $\mathcal{C}$ . Up to taking a subsequence, the sequences  $(\gamma_n \cdot a_n)_n$  and  $(\gamma_n \cdot y)_n$  and  $(\gamma_n \cdot z)_n$  converge respectively to some points  $a_\infty, y_\infty, z_\infty$  in  $\mathbb{P}(\mathbb{R}^{p,q})$ . We have  $a_\infty \in \mathcal{C}$  and  $y_\infty \in \mathcal{C} \cup \Lambda_\Gamma$  and  $z_\infty \in \Lambda_\Gamma$ . Since  $a_\infty \in [y_\infty, z_\infty] \subset z_\infty^\perp$ , the intersection of  $[y_\infty, z_\infty]$  with  $\mathbb{H}^{p,q-1}$  is contained in a lightlike geodesic, hence can meet  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  only at  $z_\infty$ . Thus  $y_\infty$  cannot belong to  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ , lest  $y_\infty = z_\infty$  and the closure of  $\mathcal{C}$  in  $\mathbb{P}(\mathbb{R}^{p,q})$  contain a full projective line, contradicting the proper convexity of  $\mathcal{C}$ . Therefore,  $y_\infty \in \mathcal{C}$ . But this contradicts the properness of the action of  $\Gamma$  on  $\mathcal{C}$ .  $\square$

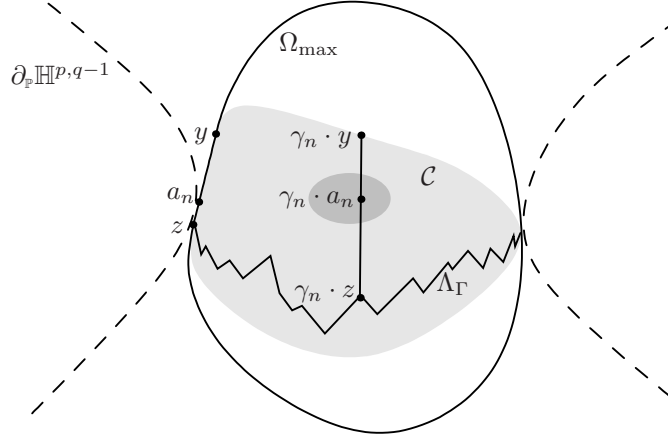


FIGURE 4. Illustration for the proof of Lemma 4.2

**4.2. Transversality of  $\Lambda_{\min}$ .** Let  $\Omega_{\min} \subset \Omega$  be the smallest nonempty,  $\Gamma$ -invariant, convex open domain of  $\mathbb{P}(\mathbb{R}^{p,q})$  (Fact 2.8 and Remark 2.9). The group  $\Gamma$  still acts properly and cocompactly on the closure  $\mathcal{C}_{\min}$  of  $\Omega_{\min}$  in  $\mathbb{H}^{p,q-1}$ , and we have  $\mathcal{C}_{\min} \subset \mathcal{C} \subset \Omega_{\max}$  by Lemma 4.2. Let  $\partial_{\mathbb{H}} \mathcal{C}_{\min} = \mathcal{C}_{\min} \setminus \Omega_{\min}$  be the boundary of  $\mathcal{C}_{\min}$  inside  $\mathbb{H}^{p,q-1}$ , and let  $\Lambda_{\min} \supset \Lambda_\Gamma$  be the intersection of  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  with the closure of  $\mathcal{C}_{\min}$  in  $\mathbb{P}(\mathbb{R}^{p,q})$ . In this section, we work with  $(\mathcal{C}_{\min}, \Lambda_{\min})$  instead of  $(\mathcal{C}, \Lambda)$ .

Let  $d = d_{\Omega_{\max}}$  be the Hilbert metric on  $\Omega_{\max}$  (see Section 2.3), and  $\partial_{\mathbb{P}}\Omega_{\max}$  the boundary of the open set  $\Omega_{\max}$  in  $\mathbb{P}(\mathbb{R}^{p,q})$ .

Let  $N$  be a maximal closed convex subset of  $\Lambda_{\min}$  and  $Q$  a maximal closed convex subset of  $\partial_{\mathbb{P}}\Omega_{\max}$  containing  $N$ . We denote by  $\partial_Q N$  and  $\partial_Q Q$  the respective boundaries of  $N$  and  $Q$  in the projective subspace spanned by  $Q$ . In particular, if  $\dim N < \dim Q$ , then  $\partial_Q N = N$ .

**Lemma 4.3.** *For any sequence  $(y_n)_{n \in \mathbb{N}}$  in  $\mathcal{C}_{\min}$  converging to a point of  $\partial_Q N \setminus \partial_Q Q$ , we have  $d(y_n, \partial_{\mathbb{H}}\mathcal{C}_{\min}) \xrightarrow{n \rightarrow +\infty} 0$ .*

*Proof.* Suppose by contradiction that this is not the case. After passing to a subsequence, there exists  $\varepsilon > 0$  such that  $d(y_n, \partial_{\mathbb{H}}\mathcal{C}_{\min}) \geq \varepsilon$  for all  $n$ . Let  $y \in \partial_Q N \setminus \partial_Q Q$  be the limit of  $(y_n)_{n \in \mathbb{N}}$  and let  $[a, b]$  be a maximal segment in  $Q$  whose intersection with  $N$  has endpoint  $y$ . As depicted in Figure 5, the straight geodesic ray from  $a$  through  $y_n$  enters  $\mathcal{C}_{\min}$  in some point  $z_n \in \mathcal{C}_{\min}$  (and then exits  $\mathcal{C}_{\min}$ ) and then hits a point  $b_n$  in  $\partial_{\mathbb{P}}\Omega_{\max}$ . The sequence  $(b_n)_{n \in \mathbb{N}}$  converges to  $b$  and  $(z_n)_{n \in \mathbb{N}}$  converges to some point  $z \in Q$  with  $d_Q(y, z)$  equal to the limit of  $d(y_n, z_n)$ , which is nonzero. (Here  $d_Q$  denotes the Hilbert metric on  $Q$ .) Hence  $z$  is a point of  $[a, y)$ , so not in  $N$ , so not in  $\Lambda_{\min}$ : contradiction.  $\square$

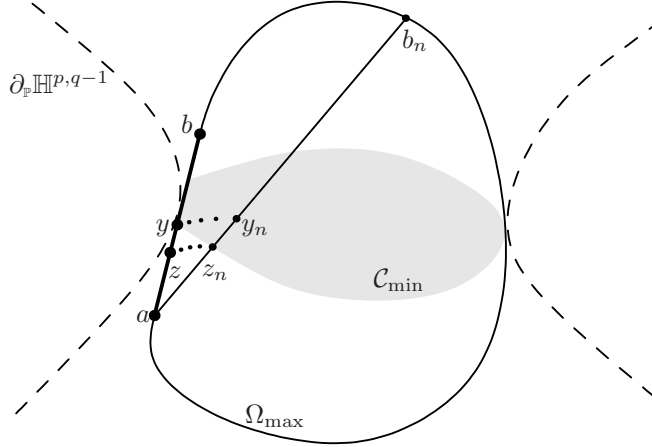


FIGURE 5. Illustration for the proof of Lemma 4.3

**Corollary 4.4.** *For  $y \in \Omega_{\min}$ , any accumulation point of the  $\Gamma$ -orbit of  $y$  in  $\partial_Q N$  is contained in  $\partial_Q Q$ .*

*Proof.* Apply Lemma 4.3 with  $(y_n)_{n \in \mathbb{N}}$  in the  $\Gamma$ -orbit of  $y$  and note that  $d(y_n, \partial_{\mathbb{H}}\mathcal{C}_{\min}) = d(y, \partial_{\mathbb{H}}\mathcal{C}_{\min}) > 0$  for all  $n$ , by  $\Gamma$ -invariance of  $d(\cdot, \cdot)$ .  $\square$

**Lemma 4.5.** *If  $N$  contains a nontrivial line segment, then it contains a nontrivial line segment which is inextendable in  $\partial_{\mathbb{P}}\Omega_{\max}$ .*



*Proof.* Suppose  $N$  contains a nontrivial line segment. To prove the lemma, it is enough to prove that  $\partial_Q N$  touches  $\partial_Q Q$  in two distinct points; the interval connecting these two points will then be a nontrivial segment of  $\Lambda_{\min}$  which cannot be extended in  $\partial_{\mathbb{P}} \Omega_{\max}$ .

Consider a point  $y \in \Omega_{\min}$ . The  $\Gamma$ -orbit of  $y$  accumulates on a nonempty subset  $R$  of  $\Lambda_{\min}$  which is closed and  $\Gamma$ -invariant, hence which contains the limit set  $\Lambda_{\Gamma}$  (see Section 2.2). In particular, the convex hull of  $R$  contains the convex hull  $\overline{\mathcal{C}_{\min}}$  of  $\Lambda_{\Gamma}$  in  $\overline{\Omega_{\max}}$ , hence every point of  $\Lambda_{\min} = \partial_i \mathcal{C}_{\min}$  lies in the convex hull of  $R$ . By Corollary 4.4, we have  $R \cap \partial_Q N \subset R \cap \partial_Q Q$ . Since  $N$  is a maximal closed convex subset of  $\Lambda_{\min}$ , every extreme point of  $\partial_Q N$  is contained in  $R$ , hence in  $\partial_Q Q$ . Since  $N$  contains a nontrivial line segment,  $\partial_Q N$  contains at least two extreme points.  $\square$

**Lemma 4.6.** *The set  $\Lambda_{\min}$  is transverse.*

*Proof.* Suppose by contradiction that  $\Lambda_{\min}$  is not transverse. This means that there are two points  $a \neq b$  in  $\Lambda_{\min}$  with  $a \in b^{\perp}$ . By convexity of  $\mathcal{C}_{\min}$ , the set  $\Lambda_{\min}$  contains the entire segment  $[a, b] \subset \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ . By Lemma 4.5, we may take  $[a, b]$  to be inextendable in  $\partial_{\mathbb{P}} \Omega_{\max}$ . Let  $c \in \Lambda_{\min}$  be such that the interior of the triangle  $\Delta abc$  contains a point of  $\mathcal{C}_{\min}$  (i.e. is not fully contained in  $\Lambda_{\min}$ ). Consider a sequence of points  $y_n$  in  $\Delta abc \cap \mathcal{C}_{\min}$  converging to an interior point of  $[a, b]$ . By the inextendability of  $[a, b]$  in  $\partial_{\mathbb{P}} \Omega_{\max}$ , the Hilbert distance from  $y_n$  to either edge  $[a, c]$  or  $[c, b]$  is tending to infinity (or equal to infinity if the edge is contained in  $\Lambda_{\min}$ ). By cocompactness, for any  $n$  there exists  $\gamma_n \in \Gamma$  such that  $\gamma_n \cdot y_n$  belongs to a fixed compact set of  $\mathcal{C}$ . Up to taking a subsequence,  $(\gamma_n \cdot y_n)_n$  converges to some  $y_{\infty} \in \mathcal{C}$ , and  $(\gamma_n \cdot a_n)_n$  and  $(\gamma_n \cdot b_n)_n$  and  $(\gamma_n \cdot c_n)_n$  converge respectively to some  $a_{\infty}, b_{\infty}, c_{\infty} \in \Lambda_{\min}$ . The triangle  $\Delta a_{\infty} b_{\infty} c_{\infty}$  is nondegenerate since it contains  $y_{\infty} \in \mathcal{C}$ . Further,  $y_{\infty}$  is infinitely far from all three edges  $[a_{\infty}, b_{\infty}]$ ,  $[b_{\infty}, c_{\infty}]$ ,  $[a_{\infty}, c_{\infty}]$  and hence all three edges are contained in  $\Lambda_{\min}$ . But this means that  $\langle a_{\infty}, b_{\infty} \rangle_{p,q} = \langle b_{\infty}, c_{\infty} \rangle_{p,q} = \langle a_{\infty}, c_{\infty} \rangle_{p,q} = 0$  and hence every point of  $\Delta a_{\infty} b_{\infty} c_{\infty}$  is null, contradicting the fact that  $y_{\infty}$  lies in  $\mathcal{C}_{\min} \subset \mathbb{H}^{p,q-1}$ .  $\square$

**Corollary 4.7.** *The set  $\Lambda_{\min}$  is negative (Definition 1.6) and equal to the limit set  $\Lambda_{\Gamma}$ .*

*Proof.* The set  $\Lambda_{\min}$  is negative by Lemmas 3.7.(2) and 4.6: it lifts to a cone  $\tilde{\Lambda}_{\min}$  of  $\mathbb{R}^{p,q} \setminus \{0\}$  on which all pairwise inner products  $\langle \cdot, \cdot \rangle_{p,q}$  of noncolinear points are negative. By definition of  $\Lambda_{\min}$ , any point  $z \in \Lambda_{\min}$  belongs to the closed convex hull of  $\Lambda_{\Gamma}$ , hence admits a lift to  $\mathbb{R}^{p,q} \setminus \{0\}$  of the form  $\sum_{i=1}^n t_i x_i$  where  $x_1, \dots, x_n \in \tilde{\Lambda}_{\min}$  project to pairwise distinct points of  $\Lambda_{\Gamma}$  and  $t_1, \dots, t_n \geq 0$ . Since  $z \in \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ , we have

$$\left\langle \sum_{i=1}^n t_i x_i, \sum_{i=1}^n t_i x_i \right\rangle_{p,q} = \sum_{1 \leq i < j \leq n} 2t_i t_j \langle x_i, x_j \rangle_{p,q} = 0,$$

and so  $n = 1$  and  $z \in \Lambda_{\Gamma}$ .  $\square$

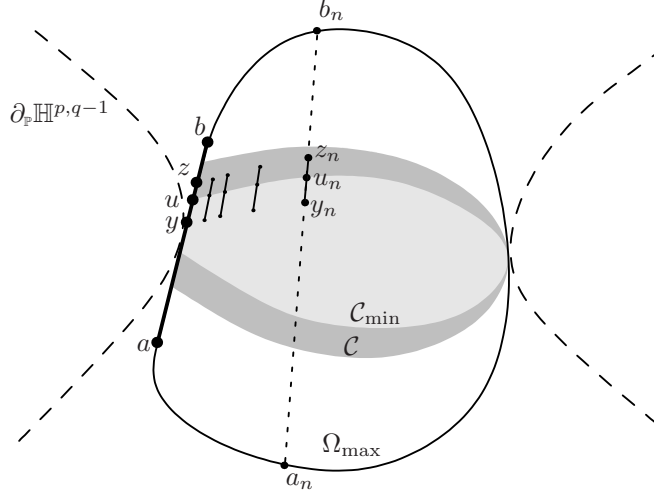


FIGURE 6. Illustration of the proof of Lemma 4.8

**4.3. Equality  $\Lambda = \Lambda_\Gamma$ .** The inclusion  $\Lambda \supset \Lambda_\Gamma$  holds by construction; we now prove that it is an equality. In particular,  $\Lambda$  is transverse by Lemma 4.6.

**Lemma 4.8.** *The ideal boundary  $\Lambda = \partial_i \mathcal{C}$  is equal to the limit set  $\Lambda_\Gamma$ .*

*Proof.* Suppose by contradiction that there exists  $z \in \Lambda \setminus \Lambda_\Gamma$ . Let  $(z_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{C} \setminus \mathcal{C}_{\min}$  converging to  $z$ . Since  $\Gamma$  is irreducible,  $\mathcal{C}_{\min}$  has nonempty interior. By cocompactness of the action of  $\Gamma$  on  $\mathcal{C}$  and  $\mathcal{C}_{\min}$ , we may find a sequence  $(y_n)_{n \in \mathbb{N}}$  in  $\mathcal{C}_{\min}$  such that  $d(y_n, z_n)$  is uniformly bounded and  $d(y_n, \partial_{\mathbb{H}} \mathcal{C}_{\min})$  is uniformly bounded away from zero. The segment  $[y_n, z_n]$  contains a unique point  $u_n$  of  $\partial_{\mathbb{H}} \mathcal{C}_{\min}$ , as depicted in Figure 6. Let  $(a_n, b_n)$  be the maximal interval of  $\Omega_{\max}$  containing  $y_n, z_n$ , so that  $d(y_n, z_n) = \frac{1}{2} \log[a_n, y_n, z_n, b_n]$  and  $d(y_n, u_n) = \frac{1}{2} \log[a_n, y_n, u_n, b_n]$ . Up to taking a subsequence, we may assume that  $a_n \rightarrow a$ ,  $b_n \rightarrow b$ ,  $u_n \rightarrow u$ , and  $y_n \rightarrow y$  where  $u$  and  $y$  belong to the line segment  $[a, b] \subset \partial_{\mathbb{P}} \Omega_{\max}$  and  $u, y \in \Lambda_\Gamma$  (Corollary 4.7). By assumption  $y \neq z$ ; since the cross ratios  $[a_n, y_n, z_n, b_n] = e^{2d(y_n, z_n)}$  are bounded away from 0, 1, and  $+\infty$ , the points  $a, y, z, b$  are pairwise distinct and  $[a_n, y_n, z_n, b_n] \rightarrow [a, y, z, b]$ . On the other hand, the cross ratios  $[a_n, y_n, u_n, b_n] = e^{2d(y_n, u_n)}$  are bounded away from 1, hence  $[a, y, u, b] \neq 1$ . Since the points  $a, y, b$  are pairwise distinct, we conclude  $y \neq u$ . But the segment  $[y, u]$  is contained in  $\partial_{\mathbb{P}} \Omega_{\max}$ , hence contained in  $\Lambda_\Gamma = \Lambda_{\min}$ , contradicting Lemma 4.6.  $\square$

**4.4. Gromov hyperbolicity of  $(\mathcal{C}, d)$ .** Using arguments inspired from [B3], we now prove that the metric space  $(\mathcal{C}, d)$  is Gromov hyperbolic with Gromov boundary  $\Lambda = \Lambda_\Gamma$ .

The following preliminary result uses the cocompactness of the action of  $\Gamma$  on  $\mathcal{C}$ , and the transversality of  $\Lambda_\Gamma$  (Lemmas 4.6 and 4.8).

**Lemma 4.9.** *There exists  $R > 0$  such that any geodesic of  $(\mathcal{C}, d)$  lies at Hausdorff distance  $\leq R$  from the projective interval with the same endpoints.*

*Proof.* Suppose by contradiction that for any  $n \in \mathbb{N}$  there is a geodesic  $\mathcal{G}_n$  with endpoints  $a_n, b_n \in \Lambda$  and a point  $y_n \in \mathcal{C}$  on that geodesic which lies at distance  $\geq n$  from the projective interval  $(a_n, b_n)$ . By cocompactness of the action of  $\Gamma$  on  $\mathcal{C}$ , for any  $n \in \mathbb{N}$  there exists  $\gamma_n \in \Gamma$  such that  $\gamma_n \cdot y_n$  belongs to a fixed compact set of  $\mathcal{C}$ . Up to taking a subsequence,  $(\gamma_n \cdot y_n)_n$  converges to some  $y_\infty \in \mathcal{C}$ , and  $(\gamma_n \cdot a_n)_n$  and  $(\gamma_n \cdot b_n)_n$  converge respectively to some  $a_\infty, b_\infty \in \Lambda$ . Since the distance from  $y_n$  to  $(a_n, b_n)$  goes to infinity, we have  $[a_\infty, b_\infty] \subset \Lambda$ , hence  $a_\infty = b_\infty$  by transversality of  $\Lambda$  (Lemmas 4.6 and 4.8). Therefore, up to extracting, the geodesics  $\mathcal{G}_n$  converge to a biinfinite geodesic of  $(\Omega_{\max}, d)$  with both endpoints equal, contradicting Lemma 2.6.  $\square$

**Lemma 4.10.** *The metric space  $(\mathcal{C}, d)$  is Gromov hyperbolic.*

*Proof.* Suppose by contradiction that triangles of  $(\mathcal{C}, d)$  are not uniformly thin. By Lemma 4.9, triangles of  $(\mathcal{C}, d)$  whose sides are projective segments are not uniformly thin: namely, there exist  $a_n, b_n, c_n \in \mathcal{C}$  and  $y_n \in [a_n, b_n]$  such that

$$(4.1) \quad d(y_n, [a_n, c_n] \cup [c_n, b_n]) \xrightarrow{n \rightarrow +\infty} +\infty.$$

By cocompactness, for any  $n$  there exists  $\gamma_n \in \Gamma$  such that  $\gamma_n \cdot y_n$  belongs to a fixed compact set of  $\mathcal{C}$ , as shown in Figure 7. Up to taking a subsequence,  $(\gamma_n \cdot y_n)_n$  converges to some  $y_\infty \in \mathcal{C}$ , and  $(\gamma_n \cdot a_n)_n$  and  $(\gamma_n \cdot b_n)_n$  and  $(\gamma_n \cdot c_n)_n$  converge respectively to some  $a_\infty, b_\infty, c_\infty \in \mathcal{C} \cup \Lambda$ . By (4.1) we have  $[a_\infty, c_\infty] \cup [c_\infty, b_\infty] \subset \Lambda$ , hence  $a_\infty = b_\infty = c_\infty$  by transversality of  $\Lambda$  (Lemmas 4.6 and 4.8). This contradicts the fact that  $y_\infty \in (a_\infty, b_\infty)$ .  $\square$

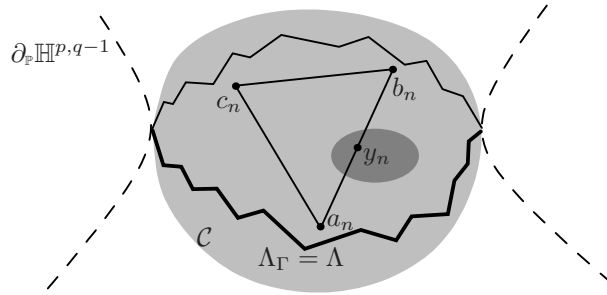


FIGURE 7. Illustration of the proof of Lemma 4.10

**Lemma 4.11.** *The Gromov boundary of  $(\mathcal{C}, d)$  is  $\Lambda = \Lambda_\Gamma$ .*

*Proof.* Fix a basepoint  $y \in \mathcal{C}$ . The Gromov boundary of  $(\mathcal{C}, d)$  is the set of equivalence classes of infinite geodesic rays in  $\mathcal{C}$  starting at  $y$ , for the equivalence relation “to remain at bounded distance for  $d$ ”. Consider the

continuous map  $\varphi$  from  $\Lambda$  to this Gromov boundary sending  $z \in \Lambda$  to the class of the geodesic ray (contained in a projective line) from  $y$  to  $z$ . This map is surjective by Lemma 4.9 and the fact that the action of  $\Gamma$  on  $\mathcal{C}$  is properly discontinuous and cocompact. Moreover, it is injective by definition of the Hilbert metric  $d$  and transversality of  $\Lambda$  (Lemmas 4.6 and 4.8).  $\square$

**4.5. Proof of the implication (i)  $\Rightarrow$  (ii) of Theorem 1.8.** As above, let  $\Gamma$  be an irreducible discrete subgroup of  $G = \mathrm{PO}(p, q)$  acting properly discontinuously and cocompactly on some nonempty, properly convex, closed subset  $\mathcal{C}$  of  $\mathbb{H}^{p, q-1}$ . By Corollary 4.7, the limit set  $\Lambda_\Gamma$  is negative. By Lemma 4.2, the closed set  $\mathcal{C}$  is contained in the largest  $\Gamma$ -invariant convex open domain  $\Omega_{\max}$  of  $\mathbb{P}(\mathbb{R}^{p, q})$ , and so we may consider the restriction to  $\mathcal{C}$  of the Hilbert metric  $d$  of  $\Omega_{\max}$ . The group  $\Gamma$  acts geometrically on the metric space  $(\mathcal{C}, d)$ , which is Gromov hyperbolic with boundary  $\Lambda_\Gamma$  by Lemmas 4.10 and 4.11. Therefore  $\Gamma$  is word hyperbolic with boundary  $\partial_\infty \Gamma = \Lambda_\Gamma$  and any orbit map  $\Gamma \rightarrow \mathcal{C}$  is a quasi-isometry, extending to a  $\Gamma$ -equivariant homeomorphism  $\xi : \partial_\infty \Gamma \rightarrow \Lambda_\Gamma$ . This homeomorphism is transverse by Lemmas 4.6 and 4.8. Since  $\Gamma$  is irreducible, we conclude (using [GW, Prop. 4.10]) that the natural inclusion  $\Gamma \hookrightarrow G = \mathrm{PO}(p, q)$  is  $P_1^{p, q}$ -Anosov.

## 5. ANOSOV SUBGROUPS WITH NEGATIVE LIMIT SET ARE $\mathbb{H}^{p, q-1}$ -CONVEX COCOMPACT

In this section we prove the implication (ii)  $\Rightarrow$  (i) of Theorem 1.8. The implication (iii)  $\Rightarrow$  (iv) of Theorem 1.8 will follow as well by Remark 4.1. Together with Proposition 1.7, this yields Theorem 1.4.(2). We also show that the connectedness assumption in Theorem 1.4.(2) cannot be removed, by providing a counterexample.

**5.1. Proof of the implication (ii)  $\Rightarrow$  (i) of Theorem 1.8.** Let  $\Gamma$  be an irreducible discrete subgroup of  $G = \mathrm{PO}(p, q)$ . Suppose that  $\Gamma$  is word hyperbolic, that the natural inclusion  $\Gamma \hookrightarrow G$  is  $P_1^{p, q}$ -Anosov, and that the limit set  $\Lambda_\Gamma \subset \partial_p \mathbb{H}^{p, q-1}$  is negative (Definition 1.6).

By Proposition 3.8, the group  $\Gamma$  preserves a nonempty properly convex open domain in  $\mathbb{P}(\mathbb{R}^{p, q})$ ; there is a largest such domain, namely

$$\Omega_{\max} := \mathbb{P}(\mathrm{Int}\{x' \in \mathbb{R}^{p, q} \mid \langle x, x' \rangle_{p, q} \leq 0 \ \forall x \in \tilde{\Lambda}_\Gamma\}),$$

where  $\tilde{\Lambda}_\Gamma$  is a cone of  $\mathbb{R}^{p, q} \setminus \{0\}$  lifting  $\Lambda$  on which  $\langle \cdot, \cdot \rangle_{p, q}$  is  $\leq 0$ ; there is also a smallest such domain  $\Omega_{\min}$ , namely the interior of the convex hull of  $\Lambda_\Gamma$  in  $\Omega_{\max}$ . By Lemma 3.7.(1) we have  $\Omega_{\min} \subset \mathbb{H}^{p, q-1}$ , and by Lemma 3.9 the closure  $\mathcal{C}_{\min}$  of  $\Omega_{\min}$  in  $\mathbb{H}^{p, q-1}$  is contained in  $\Omega_{\max}$ . In particular, the action of  $\Gamma$  on  $\mathcal{C}_{\min}$  is properly discontinuous.

To see that  $\Gamma$  is  $\mathbb{H}^{p, q-1}$ -convex cocompact and thus complete the proof of the implication (ii)  $\Rightarrow$  (i) of Theorem 1.8, it only remains to prove the following.

**Lemma 5.1.** *The action of  $\Gamma$  on  $\mathcal{C}_{\min}$  is cocompact.*

*Proof.* By Fact 1.15, the natural inclusion  $\Gamma \hookrightarrow \mathrm{PGL}(\mathbb{R}^{p,q})$  is  $P_1$ -Anosov, where  $P_1$  is the stabilizer in  $\mathrm{PGL}(\mathbb{R}^{p,q})$  of a line of  $\mathbb{R}^{p,q}$ . By [KLPa, Th. 1.7] (see also [GGKW, Rem. 5.15]), the action of  $\Gamma$  on  $\mathbb{P}(\mathbb{R}^{p,q})$  is *expanding*: for any point  $z \in \Lambda_\Gamma$  there exist an element  $\gamma \in \Gamma$ , a neighborhood  $\mathcal{U}$  of  $z$  in  $\mathbb{P}(\mathbb{R}^{p,q})$ , and a constant  $C > 1$  such that  $\gamma$  is  $C$ -expanding on  $\mathcal{U}$  for the metric

$$d_{\mathbb{P}}([x], [x']) := |\sin \angle(x, x')|$$

on  $\mathbb{P}(\mathbb{R}^{p,q})$ . We now use a version of the argument of [KLPa, Prop. 2.5], inspired by Sullivan's dynamical characterization [Su] of convex cocompactness in Riemannian hyperbolic spaces. (The argument in [KLPa] is a little more technical because it deals with bundles, whereas we work directly in  $\mathbb{P}(\mathbb{R}^{p,q})$ .)

Suppose by contradiction that the action of  $\Gamma$  on  $\mathcal{C}_{\min}$  is *not* cocompact, and let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be a sequence of positive reals converging to 0. For any  $n$ , the set  $K_n := \{z \in \mathcal{C}_{\min} \mid d_{\mathbb{P}}(z, \Lambda_\Gamma) \geq \varepsilon_n\}$  is compact, hence there exists a  $\Gamma$ -orbit contained in  $\mathcal{C}_{\min} \setminus (K_n \cup \Lambda_\Gamma)$ . By proper discontinuity of the action on  $\mathcal{C}_{\min}$ , the supremum of  $d_{\mathbb{P}}(\cdot, \Lambda_\Gamma)$  on this orbit is achieved at some point  $z_n \in \mathcal{C}_{\min}$ , and by construction  $0 < d_{\mathbb{P}}(z_n, \Lambda_\Gamma) \leq \varepsilon_n$ . Then, for all  $\gamma \in \Gamma$ ,

$$d_{\mathbb{P}}(\gamma \cdot z_n, \Lambda_\Gamma) \leq d_{\mathbb{P}}(z_n, \Lambda_\Gamma).$$

Up to extracting, we may assume that  $(z_n)_{n \in \mathbb{N}}$  converges to some  $z \in \Lambda_\Gamma$ . Consider an element  $\gamma \in \Gamma$ , a neighborhood  $\mathcal{U}$  of  $z$  in  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ , and a constant  $C > 1$  such that  $\gamma$  is  $C$ -expanding on  $\mathcal{U}$ . For any  $n \in \mathbb{N}$ , there exists  $z'_n \in \Lambda_\Gamma$  such that  $d_{\mathbb{P}}(\gamma \cdot z_n, \Lambda_\Gamma) = d_{\mathbb{P}}(\gamma \cdot z_n, \gamma \cdot z'_n)$ . We then have

$$d_{\mathbb{P}}(\gamma \cdot z_n, \Lambda_\Gamma) \geq C d_{\mathbb{P}}(z_n, z'_n) \geq C d_{\mathbb{P}}(z_n, \Lambda_\Gamma) \geq C d_{\mathbb{P}}(\gamma \cdot z_n, \Lambda_\Gamma).$$

This is impossible since  $C > 1$ .  $\square$

**5.2. Disconnected limit sets.** Let  $\Gamma$  be a free group on two generators. For  $\mathrm{rank}_{\mathbb{R}}(G) := \min(p, q) \geq 2$ , let us give an example of an irreducible  $P_1^{p,q}$ -Anosov representation  $\rho : \Gamma \rightarrow G = \mathrm{PO}(p, q)$  such that the limit set  $\Lambda_{\rho(\Gamma)}$  is neither negative nor positive (Definition 1.6). This shows that Theorem 1.4.(2) is not true when  $\partial_\infty \Gamma$  is not connected. We first work in  $\mathrm{PO}(2, 2)$  (Example 5.2), then in  $\mathrm{PO}(p, q)$  for any  $p, q \geq 2$  (Example 5.3).

**Example 5.2.** Let  $\Gamma$  be a free group on two generators. Consider two injective and discrete representations  $\rho_1, \rho_2 : \Gamma \rightarrow \mathrm{PSL}_2(\mathbb{R})$  with convex cocompact images, such that  $\rho_1$  is the holonomy of a hyperbolic 3-holed sphere and  $\rho_2$  the holonomy of a hyperbolic one-holed torus. For  $i \in \{1, 2\}$ , let  $\xi_i : \partial_\infty \Gamma \rightarrow \mathbb{P}^1 \mathbb{R}$  be the boundary map associated to  $\rho_i$ , with image  $\Lambda_i$  (a Cantor set). Let  $\psi := \xi_2 \circ \xi_1^{-1} : \Lambda_1 \rightarrow \Lambda_2$  be the unique  $(\rho_1, \rho_2)$ -equivariant homeomorphism. This map  $\psi$  does *not* preserve the cyclic order of  $\mathbb{P}^1 \mathbb{R}$ : there exist  $x_1, x_2, x_3, x_4 \in \partial_\infty \Gamma$  such that the quadruples  $(\xi_1(x_1), \xi_1(x_2), \xi_1(x_3), \xi_1(x_4))$  and  $(\xi_2(x_1), \xi_2(x_2), \xi_2(x_4), \xi_2(x_3))$  are both cyclically ordered.

The identification  $\mathrm{PO}(2, 2)_0 \simeq \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$  of Remark 2.1 lets us see  $(\rho_1, \rho_2)$  as a single representation  $\rho : \Gamma \rightarrow \mathrm{PO}(2, 2)$ . The boundary

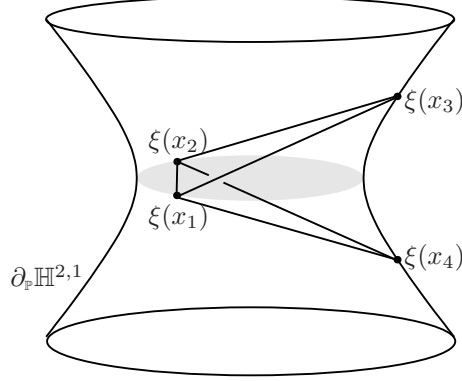


FIGURE 8. Illustration for Example 5.2. The triples  $\{\xi(x_1), \xi(x_2), \xi(x_3)\}$  and  $\{\xi(x_1), \xi(x_2), \xi(x_4)\}$  are negative, i.e. span an ideal triangle contained in  $\mathbb{H}^{2,1}$ . The triples  $\{\xi(x_1), \xi(x_3), \xi(x_4)\}$  and  $\{\xi(x_2), \xi(x_3), \xi(x_4)\}$  are positive, i.e. span ideal triangles contained in  $\mathbb{S}^{1,2}$  (which go through infinity in the picture).

maps  $\xi_1, \xi_2 : \partial_\infty \Gamma \rightarrow \mathbb{P}^1 \mathbb{R}$  associated to  $\rho_1, \rho_2$  combine into a map  $\xi$  from  $\partial_\infty \Gamma$  to the doubly ruled quadric  $\partial_{\mathbb{P}} \mathbb{H}^{2,1} \simeq \mathbb{P}^1 \mathbb{R} \times \mathbb{P}^1 \mathbb{R}$ : under this identification, the image  $\Lambda$  of  $\xi$  is the graph of  $\psi$ , and  $\rho$  is  $P_1^{2,2}$ -Anosov with boundary map  $\xi$ . However,  $\rho(\Gamma)$  is not  $\mathbb{H}^{2,1}$ -convex cocompact nor is it  $\mathbb{H}^{1,2}$ -convex cocompact (with respect to  $-\langle \cdot, \cdot \rangle_{2,2}$ ). As depicted in Figure 8, the triples  $\xi(x_1), \xi(x_2), \xi(x_3)$  and  $\xi(x_1), \xi(x_2), \xi(x_4)$  are negative, while the triples  $\xi(x_1), \xi(x_3), \xi(x_4)$  and  $\xi(x_2), \xi(x_3), \xi(x_4)$  are positive. Alternatively, observe that the six segments connecting the  $\xi(x_i)$  (for  $1 \leq i \leq 4$ ) inside  $\mathbb{H}^{2,1}$  (resp. inside  $\mathbb{S}^{1,2}$ ) carry the generator of  $\pi_1(\mathbb{P}(\mathbb{R}^{2,2}))$ , precluding the possibility that these segments could be part of a properly convex subset of  $\mathbb{P}(\mathbb{R}^{2,2})$ .

**Example 5.3.** Take any  $p, q \geq 2$  and consider the embedding

$$\tau : \mathrm{PO}(2, 2)_0 \simeq \mathrm{SO}(2, 2)_0 \hookrightarrow \mathrm{SO}(p, q)_0 \simeq \mathrm{PO}(p, q)_0$$

coming from the natural inclusion  $\mathbb{R}^{2,2} \subset \mathbb{R}^{p,q}$ . The corresponding  $\tau$ -equivariant embedding  $\iota : \partial_{\mathbb{P}} \mathbb{H}^{2,1} \hookrightarrow \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  has the property that a subset  $\Lambda$  of  $\partial_{\mathbb{P}} \mathbb{H}^{2,1}$  is negative (resp. positive) if and only if  $\iota(\Lambda)$  is negative (resp. positive) as a subset of  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ . Let  $\rho : \Gamma \rightarrow \mathrm{PO}(2, 2)_0$  be as in Example 5.2. By [GW, Prop. 4.7], the composition  $\tau \circ \rho : \Gamma \rightarrow \mathrm{PO}(p, q)_0$  is  $P_1^{p,q}$ -Anosov with limit set  $\Lambda_{\tau \circ \rho(\Gamma)} = \iota(\Lambda_{\rho(\Gamma)})$ . By Example 5.2, this limit set is neither negative nor positive. Since being Anosov is an open property [L, GW] and since a small deformation of a negative (resp. positive) triple in  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  is still negative (resp. positive), any small deformation of  $\tau \circ \rho$  in  $\mathrm{Hom}(\Gamma, \mathrm{PO}(p, q))$  is still  $P_1^{p,q}$ -Anosov representation with a limit set that is neither negative nor positive. Since  $\Gamma$  is a free group, such deformations are abundant including



many which are irreducible. The image of any such representation fails to be  $\mathbb{H}^{p,q-1}$ -convex cocompact.

## 6. LINK WITH STRONG PROJECTIVE CONVEX COCOMPACTNESS

The goal of this section is to prove Proposition 1.14. We fix an irreducible discrete subgroup  $\Gamma$  of  $G = \mathrm{PO}(p, q)$ .

**6.1. Proof of Proposition 1.14.(1).** Suppose  $\Gamma$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact, with limit set  $\Lambda_\Gamma \subset \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ . By Proposition 3.8, there is a largest  $\Gamma$ -invariant convex open domain  $\Omega_{\max}$  in  $\mathbb{P}(\mathbb{R}^{p,q})$  and a smallest nonempty  $\Gamma$ -invariant convex open domain of  $\mathbb{P}(\mathbb{R}^{p,q})$ , namely the interior  $\Omega_{\min}$  of the convex hull of  $\Lambda_\Gamma$  in  $\Omega_{\max}$ . By Corollary 4.7, the set  $\Lambda_\Gamma$  is negative. By Lemma 3.7.(1) the set  $\Omega_{\min}$  is contained in  $\mathbb{H}^{p,q-1}$ , and by Lemma 3.9 its closure  $\mathcal{C}_{\min}$  in  $\mathbb{H}^{p,q-1}$  is contained in  $\Omega_{\max}$ . In particular,  $\Gamma$  acts properly discontinuously on  $\mathcal{C}_{\min}$ . This action is cocompact since  $\Gamma$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact.

In order to prove that  $\Gamma$  is strongly projectively convex cocompact (Definition 1.13), it only remains to find a strictly convex open neighborhood  $\Omega$  of  $\mathcal{C}_{\min}$  in  $\Omega_{\max}$  with  $C^1$  boundary  $\partial_{\mathbb{P}} \Omega$ . Recall that by definition,  $\partial_{\mathbb{P}} \Omega$  is  $C^1$  if every point  $z \in \partial_{\mathbb{P}} \Omega$  has a unique *supporting hyperplane*, i.e. a unique projective hyperplane whose intersection with  $\overline{\Omega}$  is a subset of  $\partial_{\mathbb{P}} \Omega$  containing  $z$ .

**Lemma 6.1.** *The convex set  $\Omega_{\max}$  has a unique supporting hyperplane at any point  $z$  of  $\Lambda_\Gamma$ , namely  $z^\perp$ .*

*Proof.* Let  $\tilde{\Lambda}_\Gamma$  be a cone of  $\mathbb{R}^{p,q} \setminus \{0\}$  lifting  $\Lambda_\Gamma$  on which  $\langle \cdot, \cdot \rangle_{p,q}$  is nonpositive. By Proposition 3.8, the set  $\Omega_{\max}$  is the projectivization of the interior  $\tilde{\Omega}_{\max}$  of the set of  $x' \in \mathbb{R}^{p,q}$  such that  $\langle x, x' \rangle_{p,q} \leq 0$  for all  $x \in \tilde{\Lambda}_\Gamma$ . A supporting hyperplane to  $\tilde{\Omega}_{\max}$  in  $\mathbb{R}^{p,q}$  is the kernel of a linear form  $\ell = \sum_{i=1}^k \langle x_i, \cdot \rangle$  with  $x_1, \dots, x_k \in \tilde{\Lambda}_\Gamma$ . For any  $x \in \tilde{\Lambda}_\Gamma$  in such a hyperplane, we have  $\ell(x) = 0$  and  $\langle x_i, x \rangle_{p,q} \leq 0$  for all  $i$ , hence  $\langle x_i, x \rangle_{p,q} = 0$  for all  $i$ . But the set  $\Lambda_\Gamma$  is negative (Corollary 4.7), hence transverse, and so all  $x_i$  are colinear to  $x$ . Thus the unique supporting hyperplane to  $\tilde{\Omega}_{\max}$  at  $x \in \tilde{\Lambda}_\Gamma$  is  $x^\perp$ . Taking images in  $\mathbb{P}(\mathbb{R}^{p,q})$ , the unique supporting hyperplane to  $\Omega_{\max}$  at a point  $z \in \Lambda_\Gamma$  is  $z^\perp$ .  $\square$

**Lemma 6.2.** *Let  $\Omega$  be any  $\Gamma$ -invariant convex open neighborhood of  $\mathcal{C}_{\min}$  in  $\Omega_{\max}$ . Then  $\Omega$  has a unique supporting hyperplane at any point  $z$  of  $\Lambda_\Gamma$ , namely  $z^\perp$ .*

In the proof and later on, we denote by  $d$  the Hilbert metric on  $\Omega_{\max}$ .

*Proof.* For  $z \in \Lambda_\Gamma$ , let  $(y_t)_{t \geq 0}$  and  $(z_t)_{t \geq 0}$  be two geodesic rays in  $\Omega_{\max}$  with endpoint  $z$ , such that  $(z_t)_{t \geq 0}$  is contained in  $\mathcal{C}_{\min}$ . Since  $\Omega_{\max}$  has a unique supporting hyperplane at  $z$ , up to reparametrization we have  $d(y_t, z_t) \rightarrow 0$  as  $t \rightarrow +\infty$ . On the other hand, by cocompactness of the action of  $\Gamma$  on  $\mathcal{C}_{\min}$ , the open neighborhood  $\Omega$  contains some *uniform* neighborhood of  $\mathcal{C}_{\min}$  in

$(\Omega_{\max}, d)$ . Therefore, some subray  $(y_t)_{t \geq t_0}$  is contained in  $\Omega$ . This shows that  $z^\perp$  is also the unique supporting hyperplane of  $\Omega$  at  $z$ .  $\square$

For any  $r > 0$ , let  $\mathcal{U}_r$  be the open uniform  $r$ -neighborhood of  $\mathcal{C}_{\min}$  in  $(\Omega_{\max}, d)$ . It is properly convex [Bu, (18.12)]. We first make the following elementary observation.

**Lemma 6.3.** *For  $r > 0$  small enough, the open set  $\mathcal{U}_r$  is contained in  $\mathbb{H}^{p,q-1}$  and  $\Gamma$  acts properly discontinuously and cocompactly on  $\overline{\mathcal{U}_r} \setminus \Lambda_\Gamma$ .*

*Proof.* Let  $\mathcal{D} \subset \mathbb{H}^{p,q-1}$  be a compact fundamental domain for the action of  $\Gamma$  on  $\mathcal{C}_{\min}$ . The set  $\mathcal{U}_r$  is contained in  $\mathbb{H}^{p,q-1}$  whenever  $r < d(y, z)$  for all  $y \in \mathcal{D}$  and all  $z \in \Omega_{\max} \cap \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ ; we now suppose that this is the case. Let  $\mathcal{C}_r$  be the closure of  $\mathcal{U}_r$  in  $\Omega_{\max}$ ; the action of  $\Gamma$  on  $\mathcal{C}_r$  is properly discontinuous. The set  $\mathcal{C}_r$  is the union of the  $\Gamma$ -translates of the closed uniform  $r$ -neighborhood  $\mathcal{D}_r$  of  $\mathcal{D}$  in  $(\Omega_{\max}, d)$ . Since  $\mathcal{D}_r$  is compact and contained in  $\mathbb{H}^{p,q-1}$ , we have  $\mathcal{C}_r \subset \mathbb{H}^{p,q-1}$  and the action of  $\Gamma$  on  $\mathcal{C}_r$  is cocompact. By Lemma 4.8, we have  $\mathcal{C}_r = \overline{\mathcal{U}_r} \setminus \Lambda_\Gamma$ .  $\square$

**Lemma 6.4.** *For any  $r > 0$  there is a  $\Gamma$ -invariant open neighborhood  $\Omega$  of  $\mathcal{C}_{\min}$  in  $\mathcal{U}_r$  which is strictly convex with  $C^1$  boundary  $\partial_{\mathbb{P}} \Omega$ .*

Note that if  $\Omega$  is any  $\Gamma$ -invariant properly convex open neighborhood of  $\mathcal{C}_{\min}$  in  $\Omega_{\max}$ , then every point of  $\Lambda_\Gamma$  is an extreme point of  $\overline{\Omega}$  (because  $\Lambda_\Gamma$  is negative, hence transverse) with unique supporting hyperplane (Lemma 6.2). Therefore, in order to prove Lemma 6.4, we only need to focus on  $\partial_{\mathbb{P}} \Omega \setminus \Lambda_\Gamma$ , that is we must construct  $\Omega$  such that each point of  $\partial_{\mathbb{P}} \Omega \setminus \Lambda_\Gamma$  is an extreme point with unique supporting hyperplane. Constructing such a neighborhood  $\Omega$  clearly involves arbitrary choices; here is one of many possible constructions.

*Proof of Lemma 6.4.* In this proof, we fix a finite-index subgroup  $\Gamma_0$  of  $\Gamma$  which is torsion-free; such a subgroup exists by the Selberg lemma [Se, Lem. 8].

We proceed in three steps. Firstly, we construct a  $\Gamma$ -invariant open neighborhood  $\Omega_1 \subset \mathcal{U}_r$  of  $\mathcal{C}_{\min}$  in  $\Omega_{\max}$  which has  $C^1$  boundary, but which is not necessarily strictly convex. Secondly, we construct a small deformation  $\Omega_2 \subset \mathcal{U}_r$  of  $\Omega_1$  which still has  $C^1$  boundary and is strictly convex, but which is only  $\Gamma_0$ -invariant, not necessarily  $\Gamma$ -invariant. Finally, we “average” translates  $\gamma \cdot \Omega_2$  of  $\Omega_2$  for  $\gamma \Gamma_0$  ranging over the  $\Gamma_0$ -cosets of  $\Gamma$ , and construct in this way a  $\Gamma$ -invariant open neighborhood  $\Omega \subset \mathcal{U}_r$  of  $\mathcal{C}_{\min}$  which has  $C^1$  boundary and is strictly convex.

• **Construction of  $\Omega_1$ :** Consider a compact fundamental domain  $\mathcal{D}$  for the action of  $\Gamma$  on  $\mathcal{C}_{\min}$ . The convex hull of  $\mathcal{D}$  in  $\Omega_{\max}$  is still contained in  $\mathcal{C}_{\min}$ . Let  $\mathcal{D}'$  be a closed neighborhood of this convex hull in  $\mathcal{U}_r$  which has  $C^1$  boundary  $\partial_{\mathbb{P}} \mathcal{D}'$ , and let  $\Omega_1$  be the interior of the convex hull of  $\Gamma \cdot \mathcal{D}'$  in  $\mathcal{U}_r$ . The action of  $\Gamma$  on  $\overline{\Omega_1} \setminus \Lambda_\Gamma$ , as on  $\overline{\mathcal{U}_r} \setminus \Lambda_\Gamma$ , is properly discontinuous and cocompact (Lemma 6.3), hence  $\overline{\Omega_1} \setminus \Lambda_\Gamma$  is contained in  $\Omega_{\max}$  by Lemma 4.2.

Let us check that  $\Omega_1$  has  $C^1$  boundary  $\partial_{\mathbb{P}}\Omega_1$ . We first observe that any supporting hyperplane  $\Pi_y$  to  $\Omega_1$  at a point  $y \in \partial_{\mathbb{P}}\Omega_1 \setminus \Lambda_{\Gamma}$  stays away from  $\Lambda_{\Gamma}$ : indeed, if  $\Pi_y$  contained a point  $z \in \Lambda_{\Gamma}$ , then it would be equal to the unique supporting hyperplane to  $\Omega_1$  at  $z$ , namely  $z^{\perp}$  by Lemma 6.2, contradicting  $y \in \Omega_{\max}$ . On the other hand, Lemma 4.8 applied to  $\mathcal{C} := \overline{\Omega_1} \setminus \Lambda_{\Gamma}$  implies that for any neighborhood  $\mathcal{V}$  of  $\Lambda_{\Gamma}$  in  $\mathbb{P}(\mathbb{R}^{p,q})$  and any infinite sequence of distinct elements  $\gamma_j \in \Gamma$ , the translates  $\gamma_j \cdot \mathcal{D}'$  are eventually all contained in  $\mathcal{V}$ . Therefore, in a neighborhood of  $y$ , the hypersurface  $\partial_{\mathbb{P}}\Omega_1$  coincides with the convex hull of a *finite* union of translates  $\gamma \cdot \mathcal{D}'$ , which has  $C^1$  boundary by Lemma 2.7.

• **Construction of  $\Omega_2$ :** For any  $y \in \partial_{\mathbb{P}}\Omega_1 \setminus \Lambda_{\Gamma}$ , let  $Q_y$  be the intersection of  $\overline{\Omega_1}$  with the unique supporting hyperplane  $\Pi_y$  at  $y$ . By the above observation,  $Q_y$  is a closed convex subset of  $\partial_{\mathbb{P}}\Omega_1 \setminus \Lambda_{\Gamma}$ .

We claim that  $Q_y$  is disjoint from  $\gamma \cdot Q_y = Q_{\gamma \cdot y}$  for all  $\gamma \in \Gamma_0 \setminus \{1\}$ . Indeed, if there existed  $y' \in Q_y \cap Q_{\gamma \cdot y}$ , then by uniqueness the supporting hyperplanes would satisfy  $\Pi_y = \Pi_{y'} = \Pi_{\gamma \cdot y}$ , hence  $Q_y = Q_{y'} = Q_{\gamma \cdot y} = \gamma \cdot Q_y$ . This would imply  $Q_y = \gamma^n \cdot Q_y$  for all  $n \in \mathbb{N}$ , hence  $\gamma^n \cdot y \in Q_y$  for all  $n \in \mathbb{N}$ . Using the fact that the action of  $\Gamma_0$  on  $\partial_{\mathbb{P}}\Omega_1 \setminus \Lambda_{\Gamma}$  is properly discontinuous and taking a limit, we see that  $Q_y$  would contain a point of  $\Lambda_{\Gamma}$ , which we have seen is not true. Therefore  $Q_y$  is disjoint from  $\gamma \cdot Q_y$  for all  $\gamma \in \Gamma_0 \setminus \{1\}$ .

Let  $n = p + q \geq 3$ . For any  $y \in \partial_{\mathbb{P}}\Omega_1 \setminus \Lambda_{\Gamma}$ , we choose small perturbations  $\Pi_y^1, \dots, \Pi_y^{n-1}$  of the supporting hyperplane  $\Pi_y$ , in generic position, such that each  $\Pi_y^i$  cuts off a compact region  $\mathcal{Q}_y^i \supset Q_y$  from  $\overline{\Omega_1}$ , and the union  $\mathcal{Q}_y := \bigcup_{i=1}^{n-1} \mathcal{Q}_y^i$  is disjoint from all its  $\gamma$ -translates for  $\gamma \in \Gamma_0 \setminus \{1\}$ . Since  $n = p + q$ , the intersection  $\bigcap_{i=1}^{n-1} \Pi_y^i \subset \mathbb{P}(\mathbb{R}^{p,q})$  is reduced to a singleton. In addition, we ensure that  $Q_y$  has a neighborhood  $\mathcal{Q}'_y$  contained in  $\bigcap_{i=1}^{n-1} \mathcal{Q}_y^i$ .

Since the action of  $\Gamma_0$  on  $\partial_{\mathbb{P}}\Omega_1 \setminus \Lambda_{\Gamma}$  is cocompact, there exist finitely many points  $y_1, \dots, y_m \in \partial_{\mathbb{P}}\Omega_1 \setminus \Lambda_{\Gamma}$  such that  $(\partial_{\mathbb{P}}\Omega_1 \setminus \Lambda_{\Gamma}) \subset \Gamma_0 \cdot (\mathcal{Q}'_{y_1} \cup \dots \cup \mathcal{Q}'_{y_m})$ .

We now explain, for any  $y \in \partial_{\mathbb{P}}\Omega_1 \setminus \Lambda_{\Gamma}$ , how to deform  $\Omega_1$  into a new properly convex  $\Gamma_0$ -equivariant open neighborhood of  $\mathcal{C}_{\min}$  with  $C^1$  boundary, in a way that destroys all segments in  $\mathcal{Q}'_y$ . Repeating for  $y = y_1, \dots, y_m$ , this will produce a strictly convex  $\Gamma_0$ -equivariant open neighborhood  $\Omega_2$  of  $\mathcal{C}_{\min}$  with  $C^1$  boundary  $\partial_{\mathbb{P}}\Omega_2$ .

Choose an affine chart containing  $\Omega_{\max}$ , an auxiliary Euclidean metric  $g$  on this chart, and a smooth strictly concave function  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $h(0) = 0$  and  $\frac{d}{dt}|_{t=0} h(t) = 1$  (e.g.  $h = \tanh$ ). We may assume that for every  $1 \leq i \leq n-1$  the  $g$ -orthogonal projection  $\pi_y^i$  onto  $\Pi_y^i$  satisfies  $\pi_y^i(\mathcal{Q}_y^i) \subset \Pi_y^i \cap \Omega_1$ . Define maps  $\varphi_y^i : \mathcal{Q}_y^i \rightarrow \mathcal{Q}_y^i$  by the property that  $\varphi_y^i$  preserves each fiber  $(\pi_y^i)^{-1}(y')$  (a segment), taking the point at distance  $t$  from  $y'$  to the point at distance  $h(t)$ . Then  $\varphi_y^i$  takes any segment  $\sigma$  of  $Q_y$  to a strictly convex curve, unless  $\sigma$  is parallel to  $\Pi_i$ . Extending  $\varphi_y^i$  by the identity on  $\mathcal{Q}_y \setminus \mathcal{Q}_y^i$  and repeating with varying  $i$ , we find that the composition  $\varphi_y := \varphi_y^1 \circ \dots \circ \varphi_y^{n-1}$ , defined on  $\mathcal{Q}_y$ , sends all segments of  $\mathcal{Q}'_y$  to strictly convex curves. We can

extend  $\varphi_y$  in a  $\Gamma_0$ -equivariant fashion to  $\Gamma_0 \cdot \mathcal{Q}_y$ , and extend it further by the identity on the rest of  $\Omega_1$ : the set  $\varphi_y(\Omega_1)$  is still  $\Gamma_0$ -invariant, with  $C^1$  boundary, and is still contained in  $\mathcal{U}_r$ .

Repeating with finitely many points  $y_1, \dots, y_m$  as above, we obtain a strictly convex,  $\Gamma_0$ -invariant open neighborhood  $\Omega_2 \subset \mathcal{U}_r$  of  $\mathcal{C}_{\min}$  with  $C^1$  boundary  $\partial_{\mathbb{P}}\Omega_2$ .

• **Construction of  $\Omega$ :** Consider the finitely many  $\Gamma_0$ -cosets  $\gamma_1\Gamma_0, \dots, \gamma_k\Gamma_0$  of  $\Gamma$  and the corresponding translates  $\Omega'_i := \gamma_i \cdot \Omega_2$ . Let  $\Omega''$  be a  $\Gamma$ -invariant properly convex (not necessarily strictly convex) open neighborhood of  $\mathcal{C}_{\min}$  in  $\mathcal{U}_r$  which has  $C^1$  boundary  $\partial_{\mathbb{P}}\Omega''$  and is contained in all  $\Omega'_i$ ,  $1 \leq i \leq k$ . (Such a neighborhood  $\Omega''$  can be constructed for instance by the same method as for  $\Omega_1$  above.) Since  $\Omega'_i$  is strictly convex, uniform neighborhoods of  $\Omega''$  in  $(\Omega'_i, d_{\Omega'_i})$  are strictly convex [Bu, (18.12)]. Therefore, by cocompactness, if  $h : [0, 1] \rightarrow [0, 1]$  is a convex function with sufficiently fast growth (e.g.  $h(t) = t^\alpha$  for large enough  $\alpha > 0$ ), then the  $\Gamma_0$ -invariant function  $H_i := h \circ d_{\Omega'_i}(\cdot, \Omega'')$  is convex on the convex region  $H_i^{-1}([0, 1])$ , and in fact smooth and strictly convex near every point outside  $\Omega''$ . The function  $H := \sum_{i=1}^k H_i$  is  $\Gamma$ -invariant and its sublevel set  $\Omega := H^{-1}([0, 1])$  is a  $\Gamma$ -invariant open neighborhood of  $\mathcal{C}_{\min}$  in  $\mathcal{U}_r$  which is strictly convex with  $C^1$  boundary  $\partial_{\mathbb{P}}\Omega$ .  $\square$

**6.2. Proof of Proposition 1.14.(2).** Suppose  $\Gamma$  is strongly projectively convex cocompact (Definition 1.13). By Proposition 3.8, the limit set  $\Lambda_\Gamma \subset \partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$  is nonpositive or nonnegative (Definition 3.1). Moreover,  $\Lambda_\Gamma$  is transverse, because it is contained in the boundary of the strictly convex open domain of  $\mathbb{P}(\mathbb{R}^{p,q})$  preserved by  $\Gamma$  as part of the definition of strong projective convex cocompactness. Therefore,  $\Lambda_\Gamma$  is negative or positive.

Suppose  $\Lambda_\Gamma$  is negative. By Fact 2.8 and Remark 2.9, there is a smallest nonempty  $\Gamma$ -invariant convex open domain  $\Omega_{\min}$  of  $\mathbb{P}(\mathbb{R}^{p,q})$ , and a largest one  $\Omega_{\max}$ . By Lemma 3.7.(1) we have  $\Omega_{\min} \subset \mathbb{H}^{p,q-1}$ , and by Lemma 3.9 the closure  $\mathcal{C}_{\min}$  of  $\Omega_{\min}$  in  $\mathbb{H}^{p,q-1}$  is contained in  $\Omega_{\max}$ . In particular,  $\Gamma$  acts properly discontinuously on  $\mathcal{C}_{\min}$ . The quotient is compact by the strong projective convex cocompactness assumption. Thus  $\Gamma$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact.

Similarly, if  $\Lambda_\Gamma$  is positive, then its image under the natural isomorphism  $\text{PO}(p, q) \simeq \text{PO}(q, p)$  is  $\mathbb{H}^{q,p-1}$ -convex cocompact.

## 7. EXAMPLES OF $\mathbb{H}^{p,q-1}$ -CONVEX COCOMPACT SUBGROUPS

In this section we consider the following general construction.

**Proposition 7.1.** *Let  $H$  be a real semisimple Lie group of real rank 1 and  $\tau : H \rightarrow \text{SO}(p, q)$  a linear representation which is proximal, in the sense that  $\tau(H)$  contains an element which is proximal in  $\partial_{\mathbb{P}}\mathbb{H}^{p,q-1}$ . Then for any word hyperbolic group  $\Gamma$  and any convex cocompact representation  $\sigma_0 : \Gamma \rightarrow H$ ,*

- (1) the composition  $\rho_0 := \tau \circ \sigma_0 : \Gamma \rightarrow \mathrm{SO}(p, q)$  is  $P_1^{p,q}$ -Anosov and the limit set  $\Lambda_{\rho_0(\Gamma)} \subset \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  is negative or positive;
- (2) the connected component  $\mathcal{T}_{\Gamma, \rho_0}$  of  $\rho_0$  in the space of  $P_1^{p,q}$ -Anosov representations from  $\Gamma$  to  $\mathrm{SO}(p, q)$  is a neighborhood of  $\rho_0$  in  $\mathrm{Hom}(\Gamma, \mathrm{SO}(p, q))$  consisting entirely of  $P_1^{p,q}$ -Anosov representations with negative limit set or entirely of  $P_1^{p,q}$ -Anosov representations with positive limit set.

As in Section 1.6, we say that a subgroup of  $\mathrm{SO}(p, q)$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact if its image in  $\mathrm{PO}(p, q) = \mathrm{O}(p, q)/\{\pm I\}$  is.

*Proof.* Since  $H$  has real rank 1, the convex cocompact representation  $\sigma_0$  is  $P$ -Anosov where  $P$  is a minimal parabolic subgroup of  $H$  [GW, Th. 5.15]; in particular, there is an injective, continuous,  $\sigma_0$ -equivariant boundary map  $\xi_{\sigma_0} : \partial_{\infty} \Gamma \rightarrow H/P$ . By [GW, Prop. 4.7] (see also [L, Prop. 3.1]), since  $\tau$  is proximal, there is a  $\tau$ -equivariant embedding  $\iota : H/P \hookrightarrow \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  and  $\rho_0 = \tau \circ \sigma_0$  is  $P_1^{p,q}$ -Anosov with boundary map  $\iota \circ \xi_{\sigma_0} : \partial_{\infty} \Gamma \rightarrow \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ . In particular, the limit set  $\Lambda_{\rho_0(\Gamma)} = \iota \circ \xi_{\sigma_0}(\partial_{\infty} \Gamma)$  is contained in  $\Lambda := \iota(H/P)$ , which is a closed, connected subset of  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ . If  $\sigma_0(\Gamma)$  is a uniform lattice in  $H$ , then  $\Lambda_{\rho_0(\Gamma)} = \Lambda$ ; since uniform lattices of  $H$  exist, we deduce that  $\Lambda$  is transverse. By Proposition 1.7, the set  $\Lambda$  is negative or positive. In particular, for arbitrary  $\sigma_0(\Gamma)$  (not necessarily a uniform lattice), the set  $\Lambda_{\rho_0(\Gamma)} \subset \Lambda$  is negative or positive, proving (1).

Statement (2) follows from (1) and from Proposition 3.6.  $\square$

Here is an immediate consequence of Theorem 1.8, Proposition 1.14, and Proposition 7.1.

**Corollary 7.2.** *In the setting of Proposition 7.1, the group  $\rho(\Gamma)$  is strongly projectively convex cocompact (Definition 1.13) for all irreducible  $\rho \in \mathcal{T}_{\Gamma, \rho_0}$ .*

*More precisely, either  $\rho(\Gamma)$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact for all irreducible  $\rho \in \mathcal{T}_{\Gamma, \rho_0}$ , or  $\rho(\Gamma)$  is  $\mathbb{H}^{q,p-1}$ -convex cocompact (after identifying  $\mathrm{SO}(p, q)$  with  $\mathrm{SO}(q, p)$ ) for all irreducible  $\rho \in \mathcal{T}_{\Gamma, \rho_0}$ .*

**Remark 7.3.** Although in this paper we only discuss convex cocompactness in the irreducible case, Corollary 7.2 also holds for reducible representations  $\rho \in \mathcal{T}_{\Gamma, \rho_0}$ , for an appropriate definition of convex cocompactness: see [DGK2].

We now make explicit a few examples to which Corollary 7.2 applies.

**7.1.  $\mathbb{H}^{p,q-1}$ -quasi-Fuchsian groups.** Let  $\Gamma$  be the fundamental group of a convex cocompact (e.g. closed) hyperbolic manifold  $M$  of dimension  $m \geq 2$ , with holonomy  $\sigma_0 : \Gamma \rightarrow \mathrm{PO}(m, 1) = \mathrm{Isom}(\mathbb{H}^m)$ . The representation  $\sigma_0$  is  $P_1^{m,1}$ -Anosov [GW, Th. 5.15]. The limit set  $\Lambda_{\sigma_0(\Gamma)} \subset \partial_{\infty} \mathbb{H}^m$  is negative since any subset of  $\partial_{\infty} \mathbb{H}^m$  is.

For  $p, q \in \mathbb{N}^*$  with  $p \geq m$ , the natural embedding  $\mathbb{R}^{m,1} \hookrightarrow \mathbb{R}^{p,q}$  induces a linear representation  $\tau : H := \mathrm{SO}(m, 1) \rightarrow \mathrm{SO}(p, q)$  which is proximal, and

a  $\tau$ -equivariant embedding  $\iota : \partial_\infty \mathbb{H}^m \hookrightarrow \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ . The set  $\Lambda := \iota(\partial_\infty \mathbb{H}^m) \subset \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$  is negative by construction.

The representation  $\sigma_0$  lifts to a representation  $\tilde{\sigma}_0 : \Gamma \rightarrow H = \mathrm{SO}(m, 1)$ . Let  $\rho_0 := \tau \circ \tilde{\sigma}_0 : \Gamma \rightarrow \mathrm{SO}(p, q)$ . The limit set  $\Lambda_{\rho_0(\Gamma)} = \iota(\Lambda_{\sigma_0(\Gamma)}) \subset \Lambda$  is negative. Thus Corollary 7.2 implies the following.

**Proposition 7.4.** *The representation  $\rho_0 : \Gamma \rightarrow \mathrm{SO}(p, q)$  is  $P_1^{p,q}$ -Anosov. Let  $\mathcal{T}_{\Gamma, \rho_0}$  be the connected component of  $\rho_0$  in the space of  $P_1^{p,q}$ -Anosov representations from  $\Gamma$  to  $\mathrm{SO}(p, q)$ . Then  $\mathcal{T}_{\Gamma, \rho_0}$  is a neighborhood of  $\rho_0$  in  $\mathrm{Hom}(\Gamma, \mathrm{SO}(p, q))$  consisting entirely of  $P_1^{p,q}$ -Anosov representations with negative limit set. For any irreducible  $\rho \in \mathcal{T}_{\Gamma, \rho_0}$ , the group  $\rho(\Gamma)$  is strongly projectively convex cocompact, and more precisely  $\mathbb{H}^{p,q-1}$ -convex cocompact.*

For  $p = m + 1 = 3$  and  $q = 1$ , when the hyperbolic surface  $M$  is closed, the representation  $\rho_0 : \Gamma \rightarrow \mathrm{SO}(2, 1) \hookrightarrow \mathrm{SO}(3, 1)$  is called *Fuchsian*, and  $\mathcal{T}_{\Gamma, \rho_0}$  is the classical space of *quasi-Fuchsian* representations of  $\Gamma = \pi_1(M)$  into  $\mathrm{SO}(3, 1)$ , which Bers parametrized by the product of two copies of the Teichmüller space of  $M$ .

Suppose  $p = m$  and  $q = 2$ . The space  $\mathbb{H}^{p,1}$  is the  $(p + 1)$ -dimensional (Lorentzian) *anti-de Sitter* space  $\mathrm{AdS}^{p+1}$ . When the hyperbolic  $m$ -manifold  $M$  is closed, Proposition 7.4 follows from work of Mess [Me] (for  $p = 2$ ) and Barbot–Mérigot [BM] (for  $p \geq 3$ ). In that case  $\mathcal{T}_{\Gamma, \rho_0}$  is actually a full connected component of  $\mathrm{Hom}(\Gamma, \mathrm{SO}(p, 2))$ , by Mess [Me] (for  $p = 2$ ) and Barbot [Ba] (for  $p \geq 3$ ). The terminology *AdS quasi-Fuchsian* is used for  $\mathbb{H}^{p,2}$ -convex cocompact representations of  $\Gamma$  into  $\mathrm{SO}(p, 2)$ . For  $p = 2$ , these are exactly the elements of  $\mathcal{T}_{\Gamma, \rho_0}$ , and they are parametrized by the product of two copies of the Teichmüller space of  $M$  [Me]. For  $p \geq 3$ , it is conjectured [Ba] that any  $\mathbb{H}^{p,2}$ -convex cocompact representation of  $\Gamma$  lies in  $\mathcal{T}_{\Gamma, \rho_0}$ .

**7.2. Hitchin representations into  $\mathrm{SO}(m, m + 1)$  and  $\mathrm{SO}(m + 1, m + 1)$ , and maximal representations into  $\mathrm{SO}(p, 2)$ .** Let  $\Gamma$  be the fundamental group of a convex cocompact orientable hyperbolic surface, with holonomy  $\sigma_0 : \Gamma \rightarrow H := \mathrm{PSL}_2(\mathbb{R})$ . For  $m \in \mathbb{N}^*$  and  $p \geq m + 1$  and  $q \geq m$ , let  $\tau : \mathrm{PSL}_2(\mathbb{R}) \rightarrow \mathrm{SO}(p, q)$  be the composition of the  $(2m + 1)$ -dimensional irreducible linear representation  $\mathrm{PSL}_2(\mathbb{R}) \rightarrow \mathrm{SO}(m + 1, m)$  with the natural inclusion  $\mathrm{SO}(m + 1, m) \hookrightarrow \mathrm{SO}(p, q)$ . It is proximal, and there is a  $\tau$ -equivariant embedding  $\iota : \partial_\infty \mathbb{H}^2 \hookrightarrow \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ , whose image  $\Lambda := \iota(\partial_\infty \mathbb{H}^2)$  is negative if  $m$  is odd, and positive if  $m$  is even. Let  $\rho_0 := \tau \circ \sigma_0 : \Gamma \rightarrow \mathrm{SO}(p, q)$ . The limit set  $\Lambda_{\rho_0(\Gamma)} = \iota(\Lambda_{\sigma_0(\Gamma)}) \subset \Lambda$  is still negative if  $m$  is odd, and positive if  $m$  is even. Thus Corollary 7.2 implies the following.

**Proposition 7.5.** *The representation  $\rho_0 : \Gamma \rightarrow \mathrm{SO}(p, q)$  is  $P_1^{p,q}$ -Anosov.*

*Let  $\mathcal{T}_{\Gamma, \rho_0}$  be the connected component of  $\rho_0$  in the space of  $P_1^{p,q}$ -Anosov representations from  $\Gamma$  to  $\mathrm{SO}(p, q)$ . Then  $\mathcal{T}_{\Gamma, \rho_0}$  is a neighborhood of  $\rho_0$  in  $\mathrm{Hom}(\Gamma, \mathrm{SO}(p, q))$  consisting entirely of  $P_1^{p,q}$ -Anosov representations with*



negative limit set. For any irreducible  $\rho \in \mathcal{T}_{\Gamma, \rho_0}$ , the group  $\rho(\Gamma)$  is  $\mathbb{H}^{p, q-1}$ -convex cocompact if  $m$  is odd, and  $\mathbb{H}^{q, p-1}$ -convex cocompact if  $m$  is even. In particular,  $\rho(\Gamma)$  is strongly projectively convex cocompact by Proposition 1.14.

By [L], when  $(p, q) = (m+1, m)$  or  $(m+1, m+1)$  and when  $\Gamma$  is a closed surface group, the space  $\mathcal{T}_{\Gamma, \rho_0}$  of Proposition 7.5 is a full connected component of  $\text{Hom}(\Gamma, \text{SO}(p, q))$ , called the *Hitchin component* of  $\text{Hom}(\Gamma, \text{SO}(p, q))$ . Proposition 7.5 specializes in that case to Proposition 1.16.

By [BIW1, BIW3], when  $m = q = 2$  and  $\Gamma$  is a closed surface group, the space  $\mathcal{T}_{\Gamma, \rho_0}$  is a full connected component of  $\text{Hom}(\Gamma, \text{SO}(p, 2))$ , consisting of so-called *maximal representations*.

## 8. NEW EXAMPLES OF ANOSOV REPRESENTATIONS

In this section we use Theorem 1.8 to give new examples of Anosov representations, for any hyperbolic right-angled Coxeter group.

**8.1. Representations of Coxeter groups into orthogonal groups.** Let  $W_S$  be a Coxeter group generated by a finite set of involutions  $S = \{s_1, \dots, s_n\}$ , with presentation

$$W_S = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{i,j}} = 1 \quad \forall 1 \leq i, j \leq n \rangle$$

where  $m_{i,i} = 1$  and  $m_{i,j} = m_{j,i} \in \{2, \infty\}$  for  $i \neq j$ . Such a Coxeter group is called *right-angled*. It is said to be *irreducible* if  $S$  cannot be written as the disjoint union of two subsets  $S'$  and  $S''$  with  $m_{i,j} = 2$  for all  $s_i \in S'$  and  $s_j \in S''$ .

The following construction gives representations of  $W_S$  into orthogonal groups, and may be formulated for arbitrary Coxeter groups. It is a deformation of the well-known geometric representation due to Tits (see Krammer [Kr]). Let  $(e_1, \dots, e_n)$  be the standard basis of  $\mathbb{R}^n$  and  $B$  a symmetric bilinear form on  $\mathbb{R}^n$  satisfying

$$(8.1) \quad B(e_i, e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } m_{i,j} = 2, \\ -\alpha_{i,j} & \text{if } m_{i,j} = \infty, \end{cases}$$

where  $\alpha_{i,j} = \alpha_{j,i} \geq 1$  are any real numbers at least one. Consider the representation  $\rho : W_S \rightarrow \text{Aut}_{\mathbb{R}}(B) \subset \text{GL}_n(\mathbb{R})$  sending any generator  $s_i$  to the  $B$ -orthogonal reflection of  $\mathbb{R}^n$  with respect to  $e_i$ :

$$\rho(s_i) = (x \mapsto x - 2B(e_i, x)e_i).$$

It is possible that  $B$  is degenerate. To avoid this inconvenience, we can perturb the coefficients  $\alpha_{i,j}$  slightly and  $B$  becomes nondegenerate. Indeed,  $\det(B)$  is a polynomial in the variables  $\alpha_{i,j}$  which is not identically zero (it would take value 1 if all  $\alpha_{i,j}$  were set to zero).

**Remark 8.1.** If  $B$  is degenerate and one wishes to keep the chosen values of the  $\alpha_{i,j}$ , then one may work instead in the vector space  $\mathbb{R}^n / \text{Ker}(B)$ , where  $\text{Ker}(B)$  is the kernel of  $B$ . Note that  $B$  descends to a nondegenerate

symmetric bilinear form  $\overline{B}$  on  $\mathbb{R}^n/\text{Ker}(B)$  and the representation  $\rho$  to a representation  $\overline{\rho}$  into the general linear group of  $\mathbb{R}^n/\text{Ker}(B)$  that preserves  $\overline{B}$ . The following arguments easily transpose to this setting.

From now on, we assume that  $B$  is nondegenerate. We identify  $B$  with  $\langle \cdot, \cdot \rangle_{p,q}$  and  $\text{Aut}_{\mathbb{R}}(B)$  with  $\text{O}(p, q)$  for some  $p, q \in \mathbb{N}$ . The basis  $(e_1, \dots, e_n)$  becomes a basis  $(x_1, \dots, x_n)$  of  $\mathbb{R}^{p,q}$  with  $\langle x_i, x_j \rangle_{p,q} = B(e_i, e_j)$  for all  $i, j$ .

**8.2. Conditions for  $\mathbb{H}^{p,q-1}$ -convex cocompactness.** By work of Tits and Vinberg [V], the representation  $\rho$  is injective and discrete, and  $W_S$  acts properly discontinuously via  $\rho$  on the interior  $\tilde{\Omega}$  of the  $\rho(W_S)$ -orbit of the fundamental closed polyhedral cone

$$\tilde{\Delta} = \{v \in \mathbb{R}^{p,q} \mid \langle v, x_i \rangle_{p,q} \leq 0 \quad \forall 1 \leq i \leq n\}$$

in  $\mathbb{R}^{p,q}$ . Since  $B$  is nondegenerate,  $\tilde{\Delta}$  has nonempty interior. The elements  $\rho(s_i)$  are reflections in the faces of  $\tilde{\Delta}$ . Let  $\Omega$  be the image of  $\tilde{\Omega}$  in  $\mathbb{P}(\mathbb{R}^{p,q})$ . We shall prove the following.

**Theorem 8.2.** *In the setting of Section 8.1, suppose that  $W_S$  is infinite and irreducible and that the following conditions are both satisfied:*

- (1) *there does not exist disjoint subsets  $S', S''$  of  $S$  such that  $W_{S'}$  and  $W_{S''}$  are both infinite and commute;*
- (2) *the parameters  $\alpha_{ij}$  of (8.1), which define  $B$  and  $\rho$ , are all  $> 1$ .*

*Then  $\Omega$  is properly convex and the group  $\rho(W_S) \subset \text{Aut}_{\mathbb{R}}(B) \simeq \text{O}(p, q)$  acts properly discontinuously and cocompactly on  $\mathcal{C} := \Omega \cap \Omega^*$ , which is a closed properly convex subset of  $\mathbb{H}^{p,q-1}$ .*

Here we denote by  $\Omega^*$  the dual convex to  $\Omega$  (see Section 2.3), viewed as a subset of  $\mathbb{P}(\mathbb{R}^{p,q})$  using the nondegenerate bilinear form  $\langle \cdot, \cdot \rangle_{p,q}$  (see (8.2)).

Theorem 1.17 is an easy consequence of Theorems 1.4 and 8.2.

*Proof of Theorem 1.17.* Let  $W = W_S$  be a right-angled Coxeter group in  $n$  generators as above. Since finite groups trivially satisfy Theorem 1.17, we assume that  $W_S$  is infinite. We also assume that  $W_S$  is word hyperbolic; then condition (1) of Theorem 8.2 is clearly satisfied.

Suppose the Coxeter group  $W_S$  is irreducible. Let  $\rho : W_S \rightarrow \text{Aut}_{\mathbb{R}}(B) \simeq \text{O}(p, q)$  be as in Section 8.1, satisfying condition (2) of Theorem 8.2. Then  $\rho$  is an irreducible representation [BH]. The group  $\rho(W_S)$  is  $\mathbb{H}^{p,q-1}$ -convex cocompact by Theorem 8.2, hence  $\rho$  is  $P_1^{p,q}$ -Anosov by Theorem 1.4.(1).

Suppose the Coxeter group  $W_S$  is not irreducible. Then it admits a decomposition as a direct product  $W_S = W_{S'} \times W_{S''}$  for some partition  $S = S' \sqcup S''$  of the generating set, such that  $W_{S''}$  is finite and  $W_{S'}$  is an infinite, irreducible word hyperbolic Coxeter group, satisfying condition (1) of Theorem 8.2. Applying the above construction to  $W_{S'}$  instead of  $W_S$ , we find an irreducible representation  $\rho' : W_{S'} \rightarrow \text{O}(p', q')$  satisfying condition (2) of Theorem 8.2. The group  $\rho'(W_{S'})$  is  $\mathbb{H}^{p',q'-1}$ -convex cocompact by Theorem 8.2, hence  $\rho'$  is  $P_1^{p',q'}$ -Anosov by Theorem 1.4.(1). The composition

of  $\rho'$  with the natural projection  $W_S \rightarrow W_{S'} \simeq W_S/W_{S''}$  is also  $P_1^{p',q'}$ -Anosov since its restriction to the finite-index subgroup  $W_{S'}$  is (see [GW, Cor. 1.3]).  $\square$

In the context of our work [DGK1] on proper affine actions of right-angled Coxeter groups, the possibility that the representations  $\rho$  of this section might be  $P_1^{p,q}$ -Anosov was first suggested to us by Anna Wienhard.

**Remark 8.3.** Theorems 1.4 and 8.2 imply that condition (1) of Theorem 8.2 is also sufficient for  $W_S$  to be word hyperbolic. We thus recover Moussong's hyperbolicity criterion [Mo] in the case of right-angled Coxeter groups.

**8.3. Proof of Theorem 8.2.** Condition (2) of Theorem 8.2 ensures that the irreducible reflection group  $\rho(W_S)$  is of *negative type*, i.e. the Cartan matrix  $(2B(e_i, e_j))_{1 \leq i, j \leq n}$  has at least one negative eigenvalue. Therefore  $\Omega$  is properly convex by [V, Lem. 15]. The dual  $\Omega^*$ , seen as an open subset of  $\mathbb{P}(\mathbb{R}^{p,q})$  via  $\langle \cdot, \cdot \rangle_{p,q}$ , is given by

$$(8.2) \quad \Omega^* = \mathbb{P}(\{x' \in \mathbb{R}^{p,q} \mid \langle x, x' \rangle_{p,q} < 0 \quad \forall x \in \tilde{\Omega}\}).$$

**Lemma 8.4.** *The properly convex set  $\mathcal{C} = \Omega \cap \overline{\Omega^*}$  is nonempty.*

*Proof.* By Fact 2.8, the limit set  $\Lambda_{\rho(W_S)}$  is nonempty and contained in the respective closures  $\overline{\Omega}$  and  $\overline{\Omega^*}$  of  $\Omega$  and  $\Omega^*$  in  $\mathbb{P}(\mathbb{R}^{p,q})$ , hence in the intersection  $\overline{\Omega} \cap \overline{\Omega^*}$ , which is invariant under  $\rho(W_S)$ . Since  $\rho$  is irreducible, the interior  $\Omega \cap \Omega^* \subset \mathcal{C}$  is nonempty.  $\square$

Let  $\tilde{\Sigma}$  be the intersection of  $\tilde{\Delta}$  with its (closed) dual simplex  $\tilde{\Delta}^*$ :

$$\tilde{\Sigma} = \left\{ x = \sum_{i=1}^n t_i x_i \in \mathbb{R}^{p,q} \mid t_i \geq 0 \text{ and } \langle x, x_j \rangle_{p,q} \leq 0 \quad \forall 1 \leq i, j \leq n \right\}.$$

Clearly, the preimage  $\tilde{\mathcal{C}}$  of  $\mathcal{C}$  in  $\tilde{\Omega}$  is contained in the  $\rho(W_S)$ -orbit of  $\tilde{\Sigma}$ . Let  $\Sigma \subset \mathbb{P}(\mathbb{R}^{p,q})$  be the projectivization of  $\tilde{\Sigma} \setminus \{0\}$ .

**Lemma 8.5.** *The compact set  $\Sigma \subset \mathbb{P}(\mathbb{R}^{p,q})$  is contained in  $\mathbb{H}^{p,q-1}$ .*

*Proof.* Let  $x = \sum_{i=1}^n t_i x_i \in \tilde{\Sigma}$  where  $t_i \geq 0$  and  $\langle x, x_j \rangle_{p,q} \leq 0$  for all  $i, j$ . We have

$$\langle x, x \rangle_{p,q} = \sum_{i=1}^n t_i \langle x, x_i \rangle_{p,q} \leq 0,$$

hence  $x$  projects to a point of  $\mathbb{H}^{p,q-1} \cup \partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ . Suppose by contradiction that  $x$  projects to a point of  $\partial_{\mathbb{P}} \mathbb{H}^{p,q-1}$ , i.e.  $\sum_{i=1}^n t_i \langle x, x_i \rangle_{p,q} = 0$ . Then  $t_i \langle x, x_i \rangle_{p,q} = 0$  for all  $1 \leq i \leq n$ . Let  $I \subset \{1, \dots, n\}$  be the (nonempty) collection of indices  $k$  such that  $t_k > 0$ . For any  $k \in I$ , we have  $\langle x, x_k \rangle_{p,q} = 0$ , i.e.  $t_k = \sum_{i \in I \setminus \{k\}} t_i \alpha_{i,k}$ . Since  $\alpha_{i,k} > 1$  for all  $i, k$ , we reach a contradiction by considering  $k \in I$  such that  $t_k$  is minimal.  $\square$

**Lemma 8.6.** *If condition (1) of Theorem 8.2 is satisfied, then the stabilizer in  $W_S$  of any point of  $\Sigma$  is finite. In particular (see [V, Th. 2]), the compact set  $\Sigma$  is contained in  $\Omega$ .*

*Proof.* Let  $x = \sum_{i=1}^n t_i x_i \in \tilde{\Sigma}$  where  $t_i \geq 0$  and  $\langle x, x_j \rangle_{p,q} \leq 0$  for all  $i, j$ . The stabilizer of  $[x] \in \mathbb{P}(\mathbb{R}^{p,q})$  in  $W_S$  is the subgroup  $W_{S_x}$  generated by the subset

$$S_x := \{s_j \in S \mid \langle x, x_j \rangle_{p,q} = 0\}.$$

We aim to show  $W_{S_x}$  is finite. For this we split  $S_x$  into the disjoint union of its two subsets  $S_x^0 := \{s_j \in S_x \mid t_j = 0\}$  and  $S_x^> := \{s_j \in S_x \mid t_j > 0\}$ .

We claim that any element of  $S_x^0$  commutes with any element of  $S_x^>$ ; in particular,  $W_{S_x}$  is the direct product of its subgroups  $W_{S_x^0}$  and  $W_{S_x^>}$  generated respectively by  $S_x^0$  and  $S_x^>$ . Indeed, for any  $s_j \in S_x$  we have by definition

$$(8.3) \quad 0 = \langle x, x_j \rangle_{p,q} = \sum_{i=1}^n t_i \langle x_i, x_j \rangle_{p,q} = \sum_{s_i \in S^>} t_i \langle x_i, x_j \rangle_{p,q},$$

where

$$S^> := \{s_i \in S \mid t_i > 0\}.$$

If  $s_j \in S_x^0$ , then each term of the right-hand sum in (8.3) is nonpositive, hence must be zero. Thus for any  $s_i \in S^>$  and  $s_j \in S_x^0$  we have  $\langle x_i, x_j \rangle_{p,q} = 0$ , which means that  $s_i$  and  $s_j$  commute. Therefore  $W_{S_x} = W_{S_x^0} \times W_{S_x^>}$ .

Let us prove that  $W_{S_x^>}$  is finite. For this it is sufficient to prove that  $m_{j,k} = 2$  for all distinct  $s_j, s_k \in S_x^>$ . Suppose by contradiction that  $m_{j,k} = \infty$  for some  $s_j, s_k \in S_x^>$ . By definition, we have

$$\begin{aligned} 0 = \langle x, x_j \rangle_{p,q} &= t_j + \sum_{s_i \in S^>, s_i \neq s_j} t_i \langle x_i, x_j \rangle_{p,q} \\ &\leq t_j - \alpha_{j,k} t_k < t_j - t_k, \end{aligned}$$

where the last inequality uses condition (2) of Theorem 8.2. But similarly by considering  $\langle x, x_k \rangle_{p,q} = 0$ , we find  $t_k - t_j < 0$  which is impossible. Thus  $m_{j,k} = 2$  for all distinct  $s_j, s_k \in S_x^>$  and  $W_{S_x^>}$  is a finite group (a product of finitely many copies of  $\mathbb{Z}/2\mathbb{Z}$ ).

Next, observe that  $W_{S^>}$  is infinite. Indeed, by Lemma 8.4 we have

$$\langle x, x \rangle_{p,q} = \sum_{s_i, s_\ell \in S^>} t_i t_\ell \langle x_i, x_\ell \rangle_{p,q} < 0.$$

The diagonal terms in the sum are positive and so there must be a nonzero nondiagonal term. In other words, there are two distinct elements  $s_i, s_\ell \in S^>$  generating an infinite dihedral group. Since  $S^>$  and  $S_x^0$  are disjoint, condition (1) implies that  $W_{S_x^0}$  is finite.  $\square$

*Proof of Theorem 8.2.* Suppose conditions (1) and (2) are satisfied. Let  $\mathcal{C}' \subset \mathbb{P}(\mathbb{R}^{p,q})$  be the  $\rho(W_S)$ -orbit of  $\Sigma$ . By Lemmas 8.5 and 8.6, we have

$\mathcal{C}' \subset \mathbb{H}^{p,q-1} \cap \Omega$ . In particular, the action of  $W_S$  on  $\mathcal{C}'$  via  $\rho$  is properly discontinuous, and cocompact since  $\Sigma$  is a compact fundamental domain.

The set  $\mathcal{C}$  is nonempty by Lemma 8.4. Since  $\mathcal{C} \subset \mathcal{C}'$  and  $\mathcal{C}$  is closed in  $\Omega$ , the action of  $W_S$  on  $\mathcal{C}$  via  $\rho$  is also properly discontinuous and cocompact. The fact that  $\mathcal{C}$  is closed in  $\mathbb{H}^{p,q-1}$  is a consequence of the following lemma.  $\square$

**Lemma 8.7.** *Let  $\Gamma$  be a discrete subgroup of  $O(p, q)$  preserving a properly convex open subset  $\Omega \subset \mathbb{P}(\mathbb{R}^{p,q})$ . For any compact subset  $\mathcal{K}$  of  $\Omega \cap \mathbb{H}^{p,q-1}$ , the  $\Gamma$ -orbit of  $\mathcal{K}$  has no accumulation point in  $\mathbb{H}^{p,q-1}$ .*

*Proof.* Suppose by contradiction that there are sequences  $(y_n) \in \mathcal{K}^{\mathbb{N}}$  and  $(\gamma_n) \in \Gamma^{\mathbb{N}}$  such that the  $\gamma_n$  are pairwise distinct and  $z_n := \gamma_n \cdot y_n$  converges to some  $z \in \mathbb{H}^{p,q-1}$ . We can lift the  $y_n \in \mathbb{H}^{p,q-1}$  to unit vectors  $x_n \in \mathbb{R}^{p,q}$ : both the  $x_n$  and the  $\gamma_n \cdot x_n$  stay in a compact set. On the other hand, since  $\Gamma$  is discrete, there exists  $x \in \mathbb{R}^{p,q} \setminus \{0\}$  such that  $(\gamma_n \cdot x)_{n \in \mathbb{N}}$  leaves every compact subset of  $\mathbb{R}^{p,q}$ . The direction of  $\gamma_n \cdot x$  converges (up to taking a subsequence) to some null direction  $\ell$ . There exists  $\varepsilon > 0$  such that all segments  $[x_n - \varepsilon x, x_n + \varepsilon x] \subset \mathbb{R}^{p,q} \setminus \{0\}$  project to segments  $\sigma_n$  contained in  $\Omega$ . The images  $\gamma_n \cdot \sigma_n$ , which are again contained in  $\Omega$ , converge to the full projective line spanned by  $x$  and  $\ell$ . This contradicts the proper convexity of  $\Omega$ .  $\square$

**Remark 8.8.** In the proof of Theorem 8.2, we do not assume that the set  $\mathcal{C}' = \rho(W_S) \cdot \Sigma$  is convex. We only use that  $\mathcal{C}'$  is contained in  $\Omega$  and contains  $\mathcal{C} = \Omega \cap \overline{\Omega}^*$ . In fact, by studying the local convexity of  $\mathcal{C}'$  along its boundary faces, it is possible to show that  $\mathcal{C}'$  is convex and equal to  $\mathcal{C}$ ; this is done in a general setting by Greene–Lee–Marquis [GLM].

## APPENDIX A. CONNECTEDNESS IN THE SPACE OF UNORDERED TUPLES

The following general statement, on which Proposition 1.7 relies, is probably well known. We provide a proof for the reader's convenience.

**Fact A.1.** *Let  $\Lambda$  be a connected topological space with closed singletons. For  $k \geq 1$ , the space  $\Lambda^{(k)}$  of unordered  $k$ -tuples of pairwise distinct points of  $\Lambda$  is also connected.*

Given a finite subset  $X$  of  $\Lambda$  and a point  $x \in \Lambda \setminus X$ , we denote by  $\Lambda_x^X$  the connected component of  $\Lambda \setminus X$  containing  $x$ . It is an open subset of  $\Lambda$ , and its closure  $\overline{\Lambda_x^X}$  is contained in  $\Lambda_x^X \cup X$ .

**Lemma A.2.** *Let  $\Lambda$  be a connected topological space with closed singletons. For any  $k \geq 1$  and  $\{x_0, \dots, x_k\} \in \Lambda^{(k+1)}$ , there exists  $0 \leq i_0 < k$  such that the  $k$ -tuples  $\{x_0, \dots, x_k\} \setminus \{x_{i_0}\}$  and  $\{x_0, \dots, x_{k-1}\}$  belong to the same connected component of  $\Lambda^{(k)}$ .*

*Proof.* For  $0 \leq i < k$ , let  $X_i := \{x_0, \dots, x_{k-1}\} \setminus \{x_i\}$ . It is sufficient to prove the existence of  $0 \leq i_0 < k$  such that  $x_{i_0}$  and  $x_k$  belong to the same

connected component of  $\Lambda \setminus X_{i_0}$ , i.e.  $x_{i_0} \in \Lambda_{x_k}^{X_{i_0}}$ . We have

$$(A.1) \quad \overline{\bigcap_{0 \leq i < k} \Lambda_{x_k}^{X_i}} \subset \bigcap_{0 \leq i < k} \overline{\Lambda_{x_k}^{X_i}} \subset \bigcap_{0 \leq i < k} (\Lambda_{x_k}^{X_i} \cup X_i).$$

Suppose by contradiction that  $x_i \notin \Lambda_{x_k}^{X_i}$  for all  $0 \leq i < k$ : then  $x_i \notin \Lambda_{x_k}^{X_i} \cup X_i$ , so the right-hand intersection in (A.1) is disjoint from all  $X_i$  and can be rewritten  $\bigcap_{0 \leq i < k} \Lambda_{x_k}^{X_i}$ . This set is therefore open and (by (A.1)) closed, and contains  $x_k$  but no other  $x_i$ , contradicting the fact that  $\Lambda$  is connected.  $\square$

**Lemma A.3.** *Let  $\Lambda$  be a connected topological space with closed singletons. For any  $k \geq 1$  and  $\{x_0, \dots, x_k\} \in \Lambda^{(k+1)}$ , there exists  $1 \leq j_0 \leq k$  such that  $x_i \in \Lambda_{x_0}^{\{x_{j_0}\}}$  for all  $i \in \{1, \dots, k\} \setminus \{j_0\}$ .*

We call this property  $H_k$ , or  $H_k(x_0, \{x_1, \dots, x_k\})$  to be specific.

*Proof.* We argue by induction. Property  $H_1$  is vacuously true. Assuming  $H_{k-1}$  where  $k \geq 2$ , let us prove  $H_k$  by contradiction. We have

$$(A.2) \quad \overline{\bigcap_{1 \leq j \leq k} \Lambda_{x_0}^{\{x_j\}}} \subset \bigcap_{1 \leq j \leq k} \overline{\Lambda_{x_0}^{\{x_j\}}} \subset \bigcap_{1 \leq j \leq k} (\Lambda_{x_0}^{\{x_j\}} \cup \{x_j\}).$$

Suppose  $H_k(x_0, \{x_1, \dots, x_k\})$  fails: that is, for all  $1 \leq j \leq k$ ,

$$\{x_1, \dots, x_k\} \setminus \{x_j\} \not\subset \Lambda_{x_0}^{\{x_j\}}.$$

We claim that the right member of (A.2) then cannot contain any  $x_i$  for  $1 \leq i \leq k$ : indeed that would imply  $x_i \in \Lambda_{x_0}^{\{x_j\}}$  for all  $j \in \{1, \dots, k\} \setminus \{i\}$ , hence the above relationship would yield

$$\{x_1, \dots, x_k\} \setminus \{x_i, x_j\} \not\subset \Lambda_{x_0}^{\{x_j\}}$$

for all  $j \in \{1, \dots, k\} \setminus \{i\}$ , contradicting  $H_{k-1}(x_0, \{x_1, \dots, x_k\} \setminus \{x_i\})$ .

Therefore the right-hand side of (A.2) can be written  $\bigcap_{1 \leq j \leq k} \Lambda_{x_0}^{\{x_j\}}$ , which by (A.2) turns out to be closed. It is also open, and contains  $x_0$  but no other  $x_i$ : this contradicts connectedness of  $\Lambda$ . Therefore  $H_k$  holds.  $\square$

*Proof of Fact A.1.* We argue by induction on  $k$  to prove that  $\Lambda^{(k)}$  is connected for any connected topological space  $\Lambda$  with closed singletons. The case  $k = 1$  is obviously true. For  $k \geq 2$ , suppose that  $(\Lambda')^{(k-1)}$  is connected for any connected  $\Lambda'$  with closed singletons, and let us prove that  $\Lambda^{(k)}$  is connected for any connected  $\Lambda$  with closed singletons.

Consider  $\{x_0, \dots, x_k\} \in \Lambda^{(k+1)}$ . By Lemma A.3, up to exchanging the labels  $j_0$  and  $k$ , we have  $x_i \in \Lambda_{x_0}^{\{x_k\}}$  for all  $1 \leq i \leq k-1$ , i.e. all points  $x_0, \dots, x_{k-1}$  belong to the same connected component  $\Lambda'$  of  $\Lambda \setminus \{x_k\}$ . Since  $\Lambda'^{(k-1)}$  is connected, all  $(k-1)$ -tuples  $\{x_0, \dots, x_{k-1}\} \setminus \{x_i\}$  for  $0 \leq i < k$  belong to the same connected component of  $(\Lambda \setminus \{x_k\})^{(k-1)}$ , and so all  $k$ -tuples  $\{x_0, \dots, x_k\} \setminus \{x_i\}$  for  $0 \leq i < k$  belong to the same component of  $\Lambda^{(k)}$ . But by Lemma A.2 one of these  $k$ -tuples (for  $i = i_0$ ) belongs to the same component as  $\{x_0, \dots, x_{k-1}\}$ . Therefore all  $k$ -tuples contained

in  $\{x_0, \dots, x_k\}$  belong to the same component of  $\Lambda^{(k)}$ . This is true for all  $\{x_0, \dots, x_k\} \in \Lambda^{(k+1)}$ , hence  $\Lambda^{(k)}$  is connected.  $\square$

## REFERENCES

- [Ba] T. BARBOT, *Deformations of Fuchsian AdS representations are quasi-Fuchsian*, J. Differential Geom. 101 (2015), p. 1–46.
- [BM] T. BARBOT, Q. MÉRIGOT, *Anosov AdS representations are quasi-Fuchsian*, Groups Geom. Dyn. 6 (2012), p. 441–483.
- [B1] Y. BENOIST, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. 7 (1997), p. 1–47.
- [B2] Y. BENOIST, *Automorphismes des cônes convexes*, Invent. Math. 141 (2000), p. 149–193.
- [B3] Y. BENOIST, *Convexes divisibles I*, in *Algebraic groups and arithmetic*, Tata Inst. Fund. Res. Stud. Math. 17 (2004), p. 339–374.
- [B4] Y. BENOIST, *Convexes divisibles II*, Duke Math. J. 120 (2003), p. 97–120.
- [B5] Y. BENOIST, *Convexes divisibles III*, Ann. Sci. Éc. Norm. Sup. 38 (2005), p. 793–832.
- [B6] Y. BENOIST, *Convexes divisibles IV*, Invent. Math. 164 (2006), p. 249–278.
- [BH] Y. BENOIST, P. DE LA HARPE, *Adhérence de Zariski des groupes de Coxeter*, Compos. Math. 140 (2004), p. 1357–1366.
- [BPS] J. BOCHI, R. POTRIE, A. SAMBARINO, *Anosov representations and dominated splittings*, preprint, arXiv:1605.01742.
- [BCLS] M. BRIDGEMAN, R. D. CANARY, F. LABOURIE, A. SAMBARINO, *The pressure metric for Anosov representations*, Geom. Funct. Anal. 25 (2015), p. 1089–1179.
- [BIW1] M. BURGER, A. IOZZI, A. WIENHARD, *Surface group representations with maximal Toledo invariant*, Ann. Math. 172 (2010), p. 517–566.
- [BIW2] M. BURGER, A. IOZZI, A. WIENHARD, *Higher Teichmüller spaces: from  $SL(2, \mathbb{R})$  to other Lie groups*, Handbook of Teichmüller theory IV, p. 539–618, IRMA Lect. Math. Theor. Phys. 19, 2014.
- [BIW3] M. BURGER, A. IOZZI, A. WIENHARD, *Maximal representations and Anosov structures*, in preparation.
- [Bu] H. BUSEMANN, *The geometry of geodesics*, Academic Press Inc., New York, 1955.
- [CM] M. CRAMPON, L. MARQUIS, *Finitude géométrique en géométrie de Hilbert*, with an appendix by C. Vernicos, Ann. Inst. Fourier 64 (2014), p. 2299–2377.
- [DGK1] J. DANCIGER, F. GUÉRITAUD, F. KASSEL, *Proper affine actions for right-angled Coxeter groups*, preprint.
- [DGK2] J. DANCIGER, F. GUÉRITAUD, F. KASSEL, *Convex cocompact actions in real projective geometry*, preprint.
- [DHR] M. DYER, C. HOHLWEG, V. RIPOLL, *Imaginary cones and limit roots of infinite Coxeter groups*, Math. Z. 284 (2016), p. 715–780.
- [FK] T. FOERTSCH, A. KARLSSON, *Hilbert metrics and Minkowski norms*, J. Geom. 83 (2005), p. 22–31.
- [Go] W. M. GOLDMAN, *Convex real projective structures on surfaces*, J. Differential Geom. 31 (1990), p. 791–845.
- [GLM] R. GREENE, G.-S. LEE, L. MARQUIS, in preparation.
- [GGKW] F. GUÉRITAUD, O. GUICHARD, F. KASSEL, A. WIENHARD, *Anosov representations and proper actions*, to appear in Geom. Topol.
- [GW] O. GUICHARD, A. WIENHARD, *Anosov representations : Domains of discontinuity and applications*, Invent. Math. 190 (2012), p. 357–438.



- [Gu] Y. GUIVARC'H, *Produits de matrices aléatoires et applications aux propriétés géométriques des sous-groupes du groupe linéaire*, Ergodic Theory Dynam. Systems 10 (1990), p. 483–512.
- [JS] T. JANUSZKIEWICZ, J. ŚWIĄTKOWSKI, *Hyperbolic Coxeter groups of large dimension*, Comment. Math. Helv. 78 (2003), p. 555–583.
- [KLPa] M. KAPOVICH, B. LEEB, J. PORTI, *Morse actions of discrete groups on symmetric spaces*, preprint, arXiv:1403.7671.
- [KL Pb] M. KAPOVICH, B. LEEB, J. PORTI, *Some recent results on Anosov representations*, Transform. Groups 21 (2016), p. 1105–1121.
- [KL] B. KLEINER, B. LEEB, *Rigidity of invariant convex sets in symmetric spaces*, Invent. Math. 163 (2006), p. 657–676.
- [Kr] D. KRAMMER, *The conjugacy problem for Coxeter groups*, PhD thesis, Universiteit Utrecht, 1994, published in Groups Geom. Dyn. 3 (2009), p. 71–171.
- [L] F. LABOURIE, *Anosov flows, surface groups and curves in projective space*, Invent. Math. 165 (2006), p. 51–114.
- [Me] G. MESS, *Lorentz spacetimes of constant curvature* (1990), Geom. Dedicata 126 (2007), p. 3–45.
- [Mo] G. MOUSSONG, *Hyperbolic Coxeter groups*, PhD thesis, Ohio State University, 1987.
- [O] D. OSAJDA, *A construction of hyperbolic Coxeter groups*, Comment. Math. Helv. 88 (2013), p. 353–367.
- [Q] J.-F. QUINT, *Groupes convexes cocompacts en rang supérieur*, Geom. Dedicata 113 (2005), p. 1–19.
- [Se] A. SELBERG, *On discontinuous groups in higher-dimensional symmetric spaces* (1960), in “Collected papers”, vol. 1, p. 475–492, Springer-Verlag, Berlin, 1989.
- [Su] D. SULLIVAN, *Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. 155 (1985), p. 243–260.
- [V] E. B. VINBERG, *Discrete linear groups generated by reflections*, Math. USSR Izv. 5 (1971), p. 1083–1119.

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